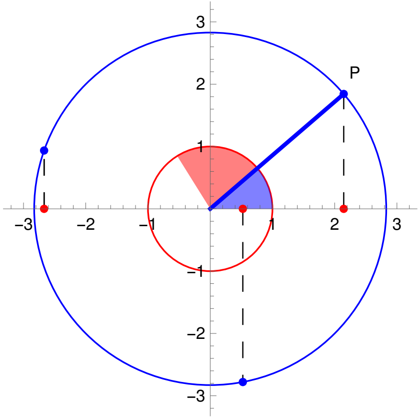
What are...solvable polynomials?

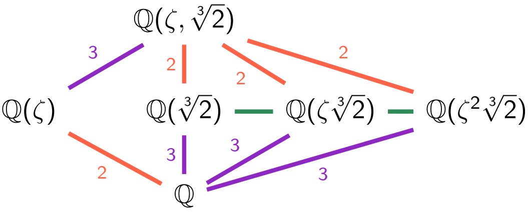
Or: Field and Galois theory, application 3

Finding explicit roots is tough



- Finding solutions to a quadratic equation is easy Completing the square
- Finding solutions to cubic or quartic equations is significantly harder
- Do we have any chance for degree ≥ 5? Well...

Back to $\mathbb{Q}(\zeta = e^{2\pi i/3}, \sqrt[3]{2})$



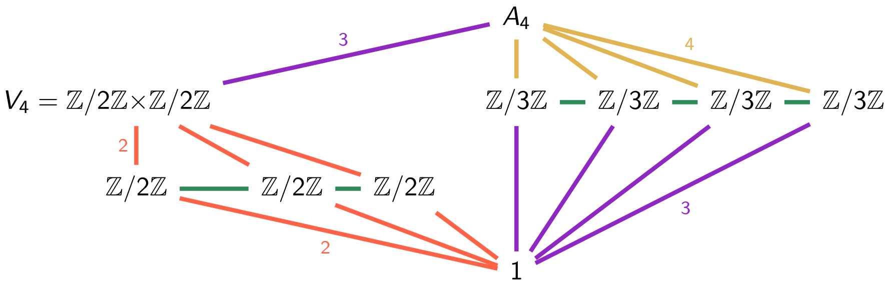
- The Galois group $G(\mathbb{Q}(\zeta, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$ is not cyclic $\mathbb{Q}$ misses ζ
- The Galois group $G(\mathbb{Q}(\zeta, \sqrt[3]{2})/\mathbb{Q}(\zeta)) \cong \mathbb{Z}/3\mathbb{Z}$ is cyclic $\mathbb{Q}(\zeta)$ has ζ

True in general! If the ground field $\mathbb{K}$ has nth roots of unity, then:

(a) The splitting field of $X^n - a$ has a cyclic Galois group

(b) For $G(\mathbb{L}/\mathbb{Q})$ cyclic $\exists a \in \mathbb{K}$ such that $\mathbb{L}$ is the splitting field of $X^n - a$

What if $G(\mathbb{L}/\mathbb{K}) \cong A_4$?



- Normal sequence $A_4 \triangleright V_4 \triangleright \mathbb{Z}/2\mathbb{Z} \triangleright 1$
- Successive cyclic quotients
  $A_4/V_4 \cong \mathbb{Z}/3\mathbb{Z}$, $V_4/\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}/1 \cong \mathbb{Z}/2\mathbb{Z}$
- Successive field extensions
  $\mathbb{Q} \subset \mathbb{Q}(2\cos(2\pi/9)) \subset \mathbb{Q}(\sqrt{-1+2\cos(2\pi/9)}) \subset \mathbb{Q}(\sqrt{-2-2\cos(2\pi/9)-2\sqrt{-1+2\cos(2\pi/9)}^{-1}})$

$2\cos(2\pi/9) = \sqrt[3]{\zeta} + \sqrt[3]{\zeta}^{-1}$, $\zeta = e^{2\pi i/3}$

For completeness: The formal statement

If $\mathbb{L}$ is the splitting field of $f \in \mathbb{Q}[X]$, then

f is solvable $\Leftrightarrow G(\mathbb{L}/\mathbb{Q})$ is solvable

- A polynomial $f \in \mathbb{K}[X]$ is solvable if $\exists$ field extensions

$$\mathbb{K} = \mathbb{K}_1 \subset \mathbb{K}_2 = \mathbb{K}_1(x_1) \subset \dots \subset \mathbb{K}_n = \mathbb{K}_{n-1}(x_{n-1}), \quad x_i \text{ root of } X^{p_i} - a_i \in \mathbb{K}_i[X]$$

with $\mathbb{K}_n$ containing the splitting field of f Adjoining p_ith roots

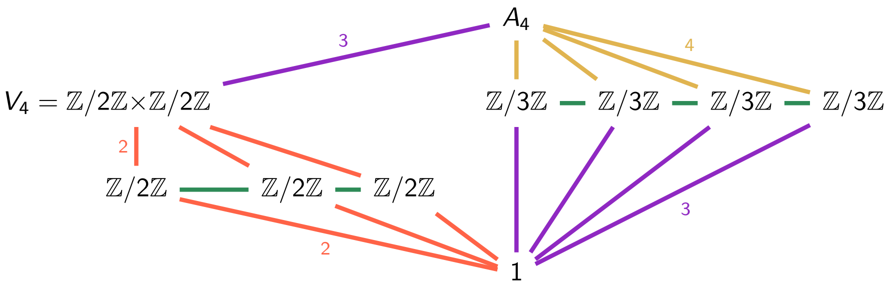
- A (finite) group G is solvable if $\exists$ normal sequence

$$G = G_1 \triangleright G_2 \triangleright \dots \triangleright G_n, \quad G_{i+1}/G_i \cong \mathbb{Z}/p_i\mathbb{Z}$$

Adjoining p_ith roots group-theoretical

Thus, almost no $f \in \mathbb{Q}[X]$ is solvable

$f = X^4 + 4\cdot X^3 + 12\cdot X^2 + 24\cdot X + 24$ **is solvable!** $G(\mathbb{L}/\mathbb{Q}) \cong A_4$



$$f = X^4 + 4\cdot X^3 + 12\cdot X^2 + 24\cdot X + 24$$

roots: $-1 \pm \sqrt{-1+2\cos(2\pi/9)} \pm \sqrt{-2\left(1+\cos(2\pi/9)+\sqrt{-1+2\cos(2\pi/9)}^{\,-1}\right)}$

- Normal sequence $A_4 \triangleright V_4 \triangleright \mathbb{Z}/2\mathbb{Z} \triangleright 1$
- Successive cyclic quotients
  $A_4/V_4 \cong \mathbb{Z}/3\mathbb{Z}$, $V_4/\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}/1 \cong \mathbb{Z}/2\mathbb{Z}$
- Successive field extensions
  $\mathbb{Q} \subset \mathbb{Q}(\cos(2\pi/9)) \subset \mathbb{Q}(\sqrt{-1+\cos(2\pi/9)}) \subset \mathbb{Q}\left(\sqrt{-2(1+\cos(2\pi/9)+\sqrt{-1+2\cos(2\pi/9)}^{\,-1})}\right)$

Thank you for your attention!

I hope that was of some help.