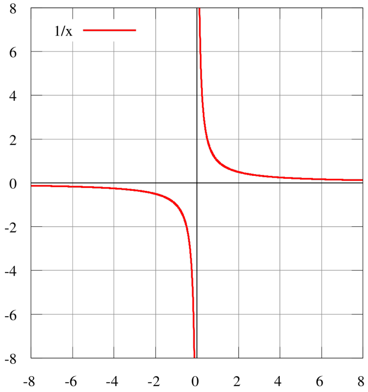
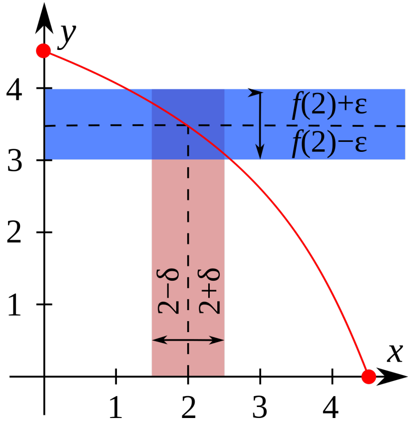
What are...regular functions?

Or: Analogs of continuous functions

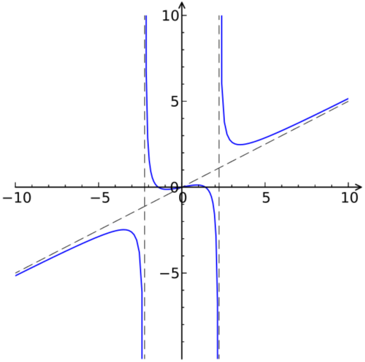
Let's mimic this!



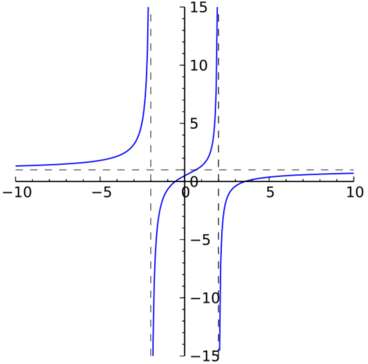


- ▶ Continuous functions play a crucial role in all of mathematics
- ▶ Two main points come to mind:
  - ▷ Some interesting continuous function are only defined on e.g. open subsets
  - ▷ Continuity is a local condition

Rational functions



Rational function of degree 3, with a graph of degree 3: $y = \frac{x^3 - 2x}{2(x^2 - 5)}$



Rational function of degree 2, with a graph of degree 3: $y = \frac{x^2 - 3x - 2}{x^2 - 4}$

- Recall that AG works with polynomials
- A natural type of maps are hence quotients f/g of two polynomials f, g
- Since $V(g)$ is closed this makes sense on open subsets of $\mathbb{K}^n$

Local versus global

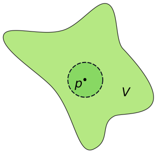
- Being a quotient is a bit of a nasty = not local condition
- Better: Assume that our maps are only quotients locally
- This mimics the locality of continuous maps

For completeness: A formal statement

Regular functions:

- V affine variety, $U \subset V$ open
- $\phi\colon U \to \mathbb{K}$ is regular if $\phi = f_p/g_p$ on U_p for all $p \in U$ for some $f_p, g_p \in \mathbb{K}[V]$

- The "for all $a \in U$" makes the condition local



- We get an important object in AG: the ring of regular functions is the $\mathbb{K}$-algebra $\mathcal{O}_V(U)$ of regular functions $\phi\colon U \to \mathbb{K}$ with pointwise operations
- What should be written (but isn't because its a mouthful) is "For every $a \in U$ there exist $f_p, g_p \in \mathbb{K}[V]$ with $g_p(x) \neq 0$ and $\phi(x) = f_p(x)/g_p(x)$ for all x in an open subset $U_p \subset U$ with $p \in U_p$"

Local is not global

```
f[x_] := (x + 3) / (x - 3);
g[x_] := (x^9 + 1) / (x - 1);
h[x_] := Piecewise[{{g[x], x < 0}, {f[x], x > 0}}];

Plot[h[x], {x, -2, 2}]
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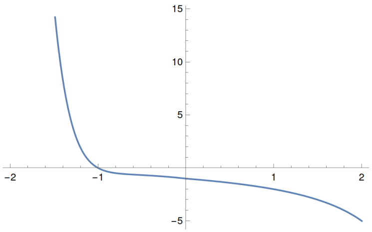

- $V = V(wx - yz)$, $U = V \setminus V(x,z) = \{(w,x,y,z) | x \neq 0 \text{ or } z \neq 0\}$ Open
- Take the regular function

$$\phi\colon U \to \mathbb{K}, (w,x,y,z) \mapsto \begin{cases} w/x & \text{if } x \neq 0, \\ y/z & \text{if } z \neq 0. \end{cases}$$

- The map ϕ is not a global quotient (look at $(0,1,0,0)$ and $(0,0,0,1)$)

Thank you for your attention!

I hope that was of some help.