What are...examples of regular functions?

Or: Regular functions and localizations

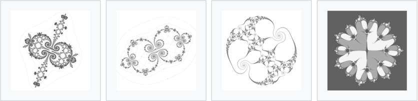
For $\mathbb{Z}$ to $\mathbb{Q}$

- $R = \mathbb{Z}$ is a ring **Numerator**
- $S = \mathbb{Z} \setminus \{0\}$ is multiplicatively closed and $1 \in S$ **Denominators**
- Every $q \in \mathbb{Q}$ is of the form $s^{-1}r$ for $r \in R$ and $s \in S$ **$\mathbb{Q} \cong S^{-1}R$**
- $(\frac{a}{b} = \frac{r}{s}) \Leftrightarrow (sa = br) \Leftrightarrow (t(sa - br) = 0$ for $t \in S)$ **Equivalence relation**
- $\mathbb{Q}$ is a ring:
  - $\mathbb{Q}$ has an addition $\frac{a}{b} + \frac{r}{s} = \frac{sa+br}{bs}$
  - $\mathbb{Q}$ has a multiplication $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$
  - $\mathbb{Q}$ has a zero $\frac{0}{1}$ and a one $\frac{1}{1}$
- $\mathbb{Z}$ is a subring of $\mathbb{Q}$ by $r \mapsto \frac{r}{1}$

Rational functions

Complex rational functions [edit]

Julia sets for rational maps



$\frac{1}{az^5+z^3+bz}$ $\frac{1}{z^3+z(-3-3i)}$ $\frac{z^2-0.2+0.7i}{z^2+0.917}$ $\frac{z^2}{z^9-z+0.025}$

- $R =$ "polynomials $\mathbb{K} \to \mathbb{K}$" is a ring **Numerator**
- $S = \{s \in R \mid s(0) \neq 0\}$ is multiplicatively closed and $1 \in S$ **Denominators**
- Every regular function L is of the form $s^{-1}r$ for $r \in R$ and $s \in S$ $L \cong S^{-1}R$
- $(\frac{a}{b} = \frac{r}{s}) \Leftrightarrow (t(sa - br) = 0 \text{ for } t \in S)$ **Equivalence relation**
- Regular functions form a ring:
  - L has an addition $\frac{a}{b} + \frac{r}{s} = \frac{sa+br}{bs}$
  - L has a multiplication $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$
  - L has a zero $\frac{0}{1}$ and a one $\frac{1}{1}$
- R has a map to L by $r \mapsto \frac{r}{1}$

Localization

Let R be a commutative ring and S be a multiplicatively closed set with $1 \in S$

(a) Equivalence relation on $R \times S$

$$(a, b) \sim (r, s) \Leftrightarrow \exists t \in S : t(sa - br) = 0$$

(b) The set of equivalence classes $S^{-1}R$ Localization (localize at S)

(c) Addition on $S^{-1}R$

$$(a, b) + (r, s) = (sa + br, bs)$$

(d) Multiplication on $S^{-1}R$

$$(a, b) \cdot (r, s) = (ar, bs)$$

Slogan. Invert elements of S

- $S^{-1}R$ is a ring with zero $(0, 1)$ and one $(1, 1)$
- There is a ring homomorphism $\iota\colon R \to S^{-1}R$ given by $r \mapsto (r, 1)$
- ι is injective (R is a subring of $S^{-1}R$) if and only if S contains no zero divisors

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- Recall Regular functions = things of the form $\phi = f/g$
- Recall Localization = the thing above
- Observation Kind of the same, right?

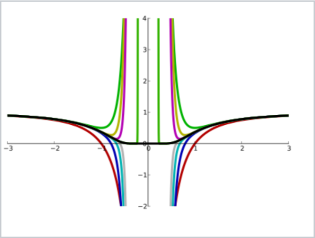
For completeness: A formal statement

For V affine variety, $f \in \mathbb{K}[V]$ we have:

$$\mathcal{O}_V(D(f)) \cong \mathbb{K}[V]_f$$

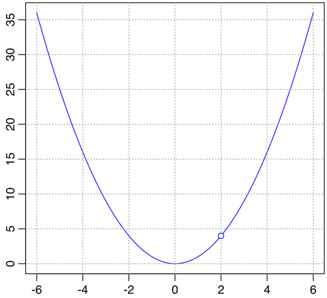
where $\mathbb{K}[V]_f =$ localization of $\mathbb{K}[V]$ along $S = \{f, f^2, f^3, \dots\}$

- Here $D(f) = V \setminus V(f) = \{v \in V | f(v) \neq 0\}$ are distinguished open sets
- Regular to coordinate functions $\leftrightsquigarrow$ Laurent to usual polynomials



e^{-1/x^2} and Laurent approximations

Complex versus algebraic geometry – again



- Example (not a distinguished open set) $V = \mathbb{K}^2$ and $U = \mathbb{K}^2 \setminus 0$, then

$$\mathcal{O}_V(U) \cong \mathbb{K}[x, y] \cong \mathcal{O}_V(V)$$

  so one can extend functions from U to V

- In complex analysis: every holomorphic function on $\mathbb{C}^2 \setminus 0$ can be extended holomorphically to $V = \mathbb{C}^2$

Thank you for your attention!

I hope that was of some help.