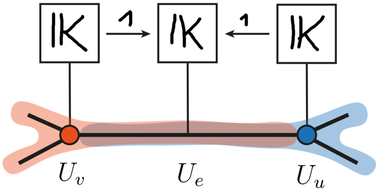
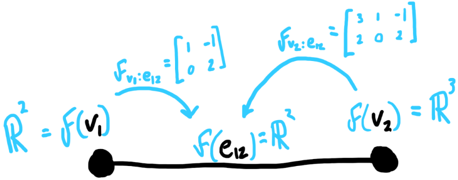
What are...examples of sheaves?

Or: Sheaves are everywhere

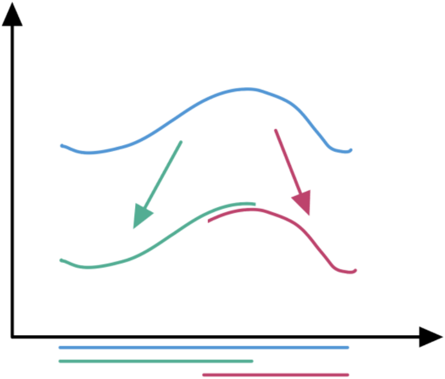
My favorite example





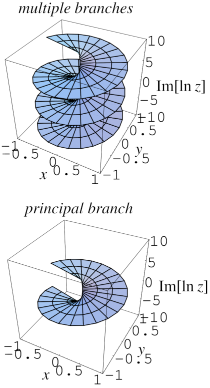
- Sheaf of fin. dim. vector spaces on X (some graph)
- $\mathcal{F}(U) = \mathbb{R}^n$ for $n = n(U)$
- $\mathcal{F}(U, V)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ = a matrix
- Variation Replace "fin. dim. vector spaces" with some more structure object

The textbook example



- Sheaf of abelian groups on X (some topological space)
- $\mathcal{F}(U) = \{f\colon U \to \mathbb{R} | f \text{ is continuous}\}$ with point-wise addition
- $\mathcal{F}(U, V)\colon \mathcal{F}(U) \to \mathcal{F}(V) = \text{restriction map}$
- Variation Replace "continuous" with some other local property

The fancy example

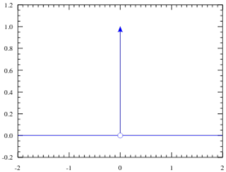


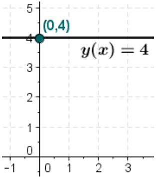
- Sheaf of sections of $f\colon Y \to X$ (some continuous map)
- $\mathcal{F}(U) = \{s\colon U \to Y | f \circ s = id_U\}$
- $\mathcal{F}(U, V)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ = restriction map
- Example $f = exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $\mathcal{F}(U)$ = branches of the complex logarithm

For completeness: A list

Here is a list of important sheaves beyond the previous examples:

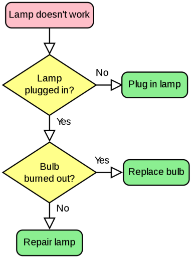
- On (smooth) manifolds
  - j times differentiable $\mathcal{O}_M^j$ or smooth $\mathcal{O}_M^\infty$ functions $M \to \mathbb{R}$
  - The one for maximal j is called the structure sheaf
  - Differential forms Ω_M^p (of degree p) on M
- Skyscraper sheaf on a topological space associated to a fixed point a fixed gadget and 'nothing' to the rest, e.g. an abelian group to x and the zero group to $y \neq x$
- Constant sheaf on a topological space associated to every point a fixed gadget, e.g. an abelian group

Skyscraper sheaf: 

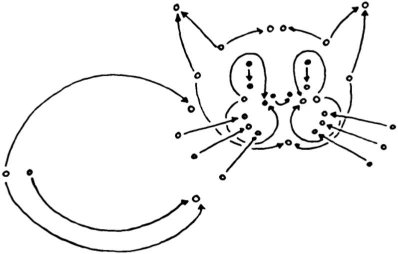
versus constant sheaf: 

The categorical "example"

Flowcharts (FC)



Categories (C)



- Presheaf categorically = a functor from the op of lattice of open sets of a topological space X to whatever you want your sheaf to be equipped with, e.g.

$$\mathbf{Open}_X^{op} \to \mathbf{Set}$$

We get that sheaf form a category $[\mathbf{Open}_X^{op}, \mathbf{Set}]$

- Sheaf categorically = add an equalizer condition
- This perspective leads to a generalization ("topos") and more examples

Thank you for your attention!

I hope that was of some help.