What are...ringed spaces?

Or: Enter, morphisms!

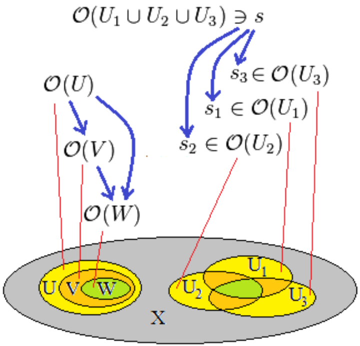
Better than spaces: relations between them

In topology a cow and a sphere are the same!

$\cong$

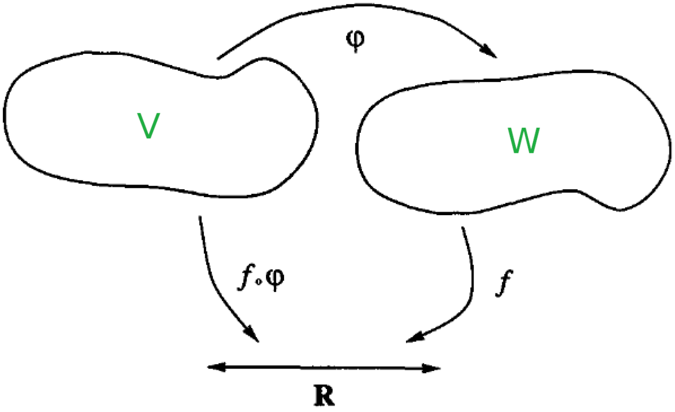
- Slogan Spaces: good; maps: better
- Example The main stars of topology are continuous maps: topologists never study $x^2 + y^2 = 1$ itself but rather the class of its continuous deformations
- Question What are maps/morphisms of sheaves?

Rings attached to open sets



- **Ringed spaces** $= X$ plus a sheaf of rings $\mathcal{O}_X$ on X (some topological space)
- **Example** An (affine) variety V is ringed using $\mathcal{O}_V =$ sheaf of regular functions (recall: regular functions are “polynomials” $V \to \mathbb{K}$)
- **Example** More generally any X with $\mathcal{O}_X =$ sheaf of functions $X \to \mathbb{K}$

Morphism of varieties



- Question What could be a morphism $\varphi\colon V \to W$ of varieties?
- Let us think of V as the pair $(V, \mathcal{O}_V)$ (V and its regular functions)
- It then makes sense to demand that the pullback $\varphi^* f = f \circ \varphi$ is a regular function for all regular functions $f\colon U \subset W \to \mathbb{K}$

For completeness: A formal statement

Assume for the time being that each sheaf consists of functions $U \to \mathbb{K}$

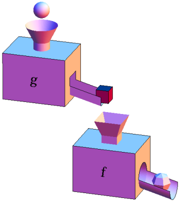
$\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces if:

(i) It is continuous

(ii) For $f \in \mathcal{O}_Y(U)$ we have $\varphi^* f \in \mathcal{O}_X(f^{-1}(U))$

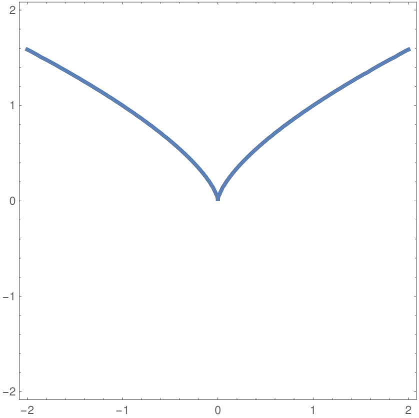
An isomorphism is a morphism with an inverse that is also a morphism

- This is a slightly silly definition: it doesn't work for general ringed spaces since $\mathcal{O}_X$ does not need to consist of maps at all (more later)
- Good Compositions and restrictions of morphisms are morphisms



Example? Absolutely!

$x^2 - y^3 = 0$:



- The above cubic V should not be isomorphic to a line (look at the singularity)
- The map $\mathbb{R} \to V, t \mapsto (t^3, t^2)$ is a morphism and bijective
- Good The inverse $(x, y) \mapsto x/y$ for $y \neq 0$ and 0 for $y = 0$ is not a morphism

Thank you for your attention!

I hope that was of some help.