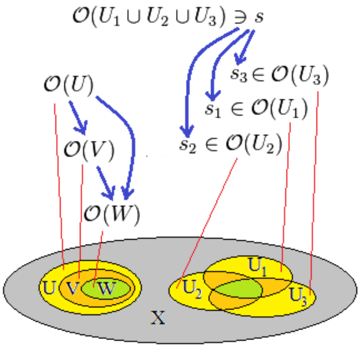
What is...the Why of ringed spaces?

Or: Geometry and algebra again

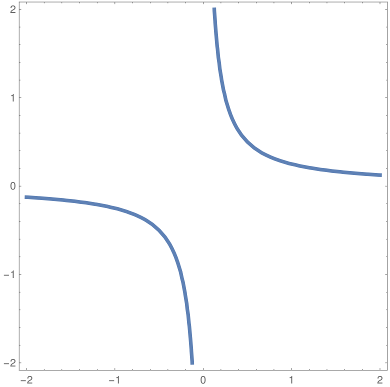
Reminder on ringed spaces



- Ringed spaces $=X$ plus a sheaf of rings $\mathcal{O}_X$ on X (some topological space)
- Example An (affine) variety V is ringed using $\mathcal{O}_V =$ sheaf of regular functions (recall: regular functions are "polynomials" $V \to \mathbb{K}$)
- Key This allowed us to define maps of varieties

From rings to morphisms – hyperbola

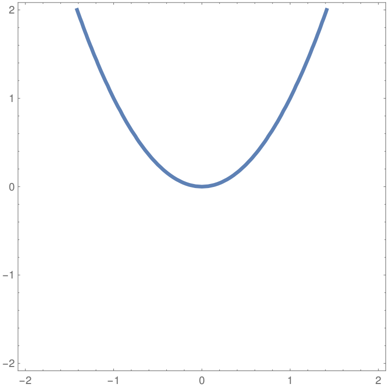
$V(xy - 1/4)$:



- Consider the conic $V = V(xy - 1)$ with $\mathbb{C}[V] = \mathbb{C}[x, y]/(xy - 1)$
- Questions How many maps (of varieties) are there $\mathbb{C} \to V$?
- Every map $f\colon \mathbb{C} \to V$ gives rise to a map $\mathbb{C}[t] \leftarrow \mathbb{C}[V]\colon f^*$, but there are no nontrivial ones

From rings to morphisms – parabola

$V(y - x^2)$:



- Consider the conic $V = V(y - x^2)$ with $\mathbb{C}[V] = \mathbb{C}[x, y]/(y - x^2)$
- Questions How many maps (of varieties) are there $\mathbb{C} \to V$?
- Every map $f\colon \mathbb{C} \to V$ gives rise to a map $\mathbb{C}[t] \leftarrow \mathbb{C}[V]\colon f^*$, and there are nontrivial ones e.g. $t \mapsto (t, t^2) \leftrightsquigarrow x \mapsto t, y \mapsto t^2$

For completeness: A formal statement

For affine varieties V, W (over alg. closed $\mathbb{K}$) there is a bijection

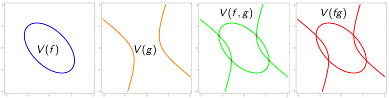
$$\{\text{maps } V \to W\} \xleftrightarrow{1:1} \{\text{homomorphisms } \mathbb{K}[W] \to \mathbb{K}[V]\}, f \mapsto f^*$$

Moreover, we have

$$\{\text{affine varieties}\}/\text{iso.} \xleftrightarrow{1:1} \{\text{fin. gen. reduced } \mathbb{K}\text{-algebras}\}/\text{iso.}$$

- Fin. gen reduced $\mathbb{K}$-algebras $\rightsquigarrow$ quotient of a polynomial ring
- This may remind you about Hilbert's Nullstellensatz

Identifying varieties and ideals



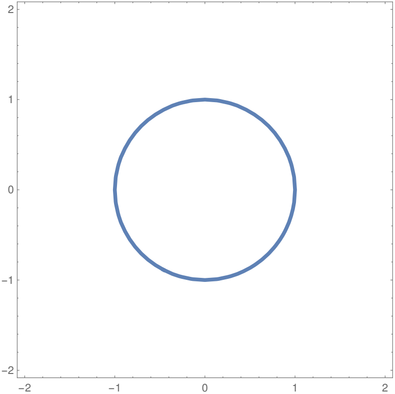
- We have bijections

$$\{\text{varieties}\} \xleftrightarrow{1:1} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \leftarrow P$$

- Radical ideal means $I = \sqrt{I}$
- One mild catch: the above are order reversing

What about the ellipse?

$x^2 + y^2 = 1$:



- Rewrite $x^2 + y^2 - 1 = 0$ as $(x + iy)(x - iy) - 1 = 0$ as $XY - 1 = 0$
- Hence, ellipse = hyperbola up to coordinate transformation over $\mathbb{C}$
- Indeed, $\mathbb{C}[x, y]/(xy - 1)$ and $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ are isomorphic
- Careful Over $\mathbb{R}$, ellipse, parabola, hyperbola are all different

Thank you for your attention!

I hope that was of some help.