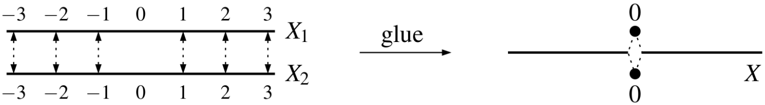
**What are...(abstract) varieties?**

Or: Zeros, once more!

# Manifolds – good and bad

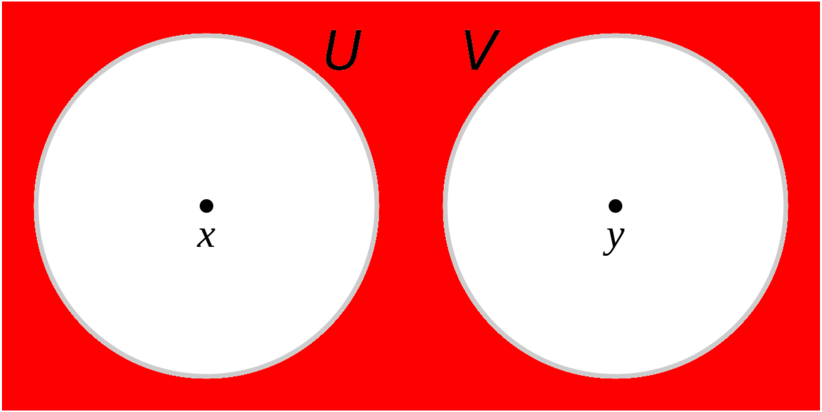
Good:

Bad:




- **Manifold** = something that is patched together from ‘discs’
- **The hope** Manifolds should be nice and pleasant spaces
- **Problem** The above definition allows pathologies

# Hausdorff vs. pathologies

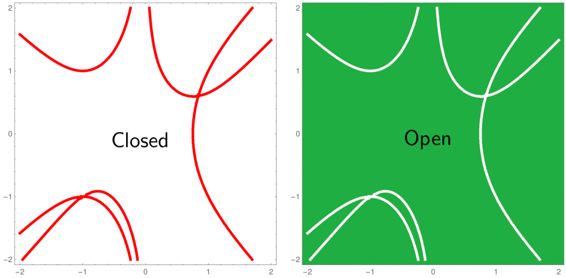



- Hausdorff = we can separate points
- Manifold (‘correct def’) = as before but also second countable Hausdorff space; the good one is a manifold, the bad one is not
- Second countable = forget for today

# Hausdorff vs. AG

The Zariski topology is a bit obscure, e.g.:

- Closed sets are very small, open sets are very large




- One cannot separate points

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- Recall We are using the Zariski topology
- Hence, spaces in AG are essentially never Hausdorff
- Better Mimic: "$X$ is Hausdorff $\Leftrightarrow \{(x,x) | x \in X\} \subset X \times X$ is closed"

## For completeness: A formal statement

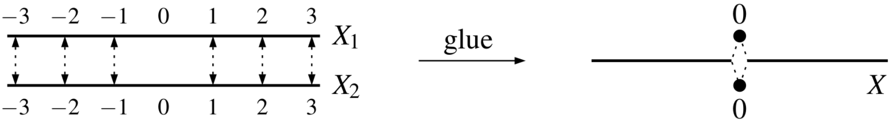
We define:

- A prevariety is a ringed space with a finite open cover by affine varieties
- A variety is a prevariety $V$ such that the diagonal

$$\Delta(V) = \{(v, v) | v \in V\} \subset V \times V$$

  is closed in $V \times V$

- Good Affine varieties are varieties but this one is not:




- Essentially everything we have seen for affine varieties works in this generality
- Careful The product of varieties is not the topological product (next slide)

# Products in AG

$$V = \mathbb{K} \Rightarrow \mathbb{K}[V] = \mathbb{K}[x]$$
$$W = \mathbb{K} \Rightarrow \mathbb{K}[W] = \mathbb{K}[y]$$
$$V \times W \cong \mathbb{K}^2 \Rightarrow \mathbb{K}[V \times W] = \mathbb{K}[x, y]$$

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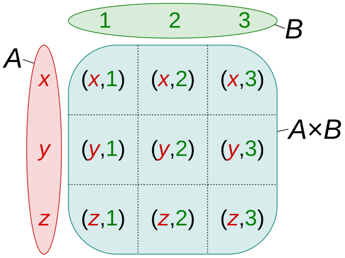


- For affine varieties $V \subset \mathbb{K}^m, W \subset \mathbb{K}^n$ the product $V \times W \subset \mathbb{K}^{m+n}$ is defined to have the Zariski topology not the product topology
- We just mimic that for prevarieties (formally: universal property)
- This is the 'correct' definition:
  - This does what it should do in all examples (e.g. as above)
  - This is the categorical product (universal property)
  - On the coordinate rings this works as well: $\mathbb{K}[V \times W] \cong \mathbb{K}[V] \otimes \mathbb{K}[W]$

**Thank you for your attention!**

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I hope that was of some help.