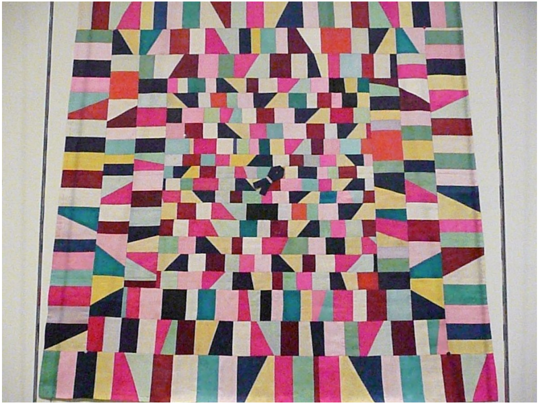
What is...projective space?

Or: Let's meet at infinity

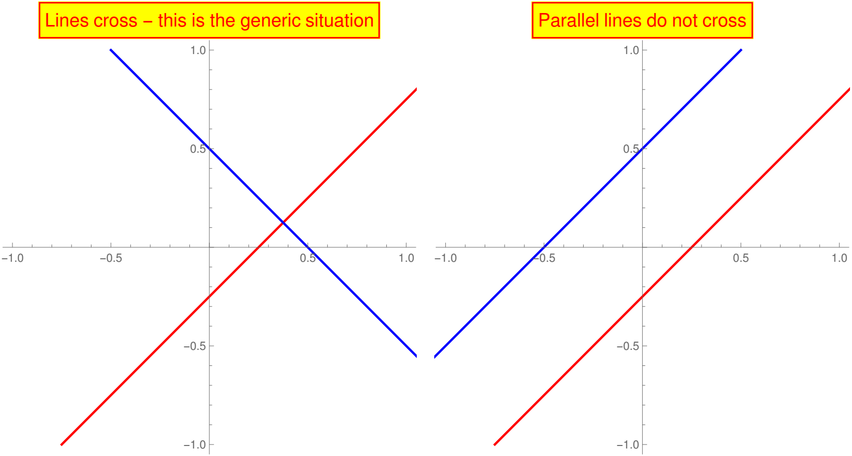
Compact varieties

A variety:



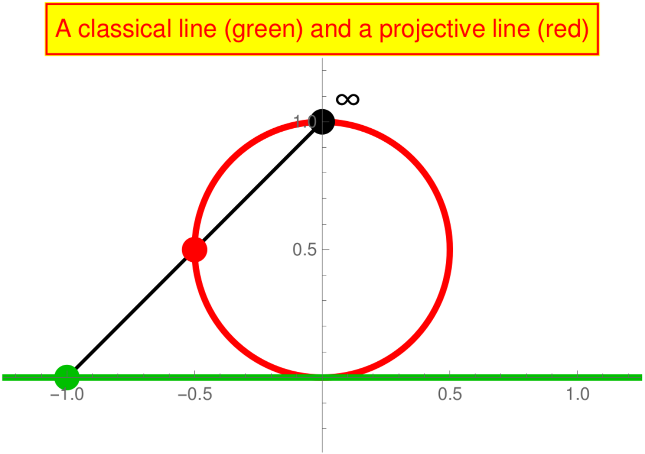
- Recall Affine varieties (zero sets in affine space $\mathbb{K}^n$) are rarely compact and we thus introduced the idea of 'patchworks'
- A large class of compact varieties is formed by projective varieties (zero sets in projective space $\mathbb{P}^n$), and these are easier than general 'patchworks'
- This time What actually is projective space $\mathbb{P}^n$?

Lines should always cross!



- Annoyance in classical = affine geometry: lines need not to cross
- This is however not wrong by a lot (only parallels do not cross)
- Projective geometry fixes this annoyance

Adding infinity

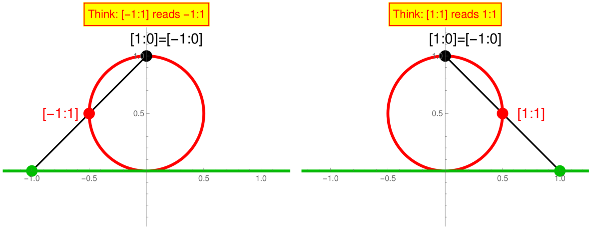


- Above, the green point corresponds to the red point
- Bottom (green): a classical line ($\mathbb{R}$, not compact)
- Bottom (red): a projective line (a sphere, compact)

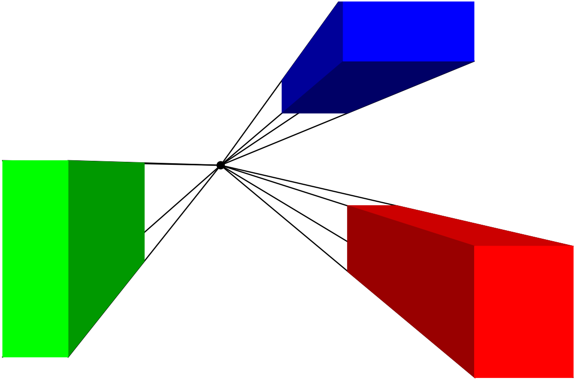
For completeness: A formal statement

Projective space $\mathbb{P}^n$ is the set of equivalence classes of $\mathbb{K}^{n+1} \setminus \{0\}$ under
$$v \sim w \Leftrightarrow v = \lambda w.$$

- Homogeneous (projective) coordinates If $v = x_0 v_0 + ... + x_n v_n$ in a basis $\{v_0, ..., v_n\}$ of $\mathbb{K}^{n+1}$, then $[x_0 : ... : x_n]$ are the projective coordinates
- $[x_0 : ... : x_n] = [\lambda x_0 : ... : \lambda x_n]$ (Vector space up to scaling)
- $[x_0 : ... : x_{n-1} : 0]$ are points at infinity
- The points $[x_0 : ... : x_{n-1} : x_n \neq 0]$ are classical points via $[x_0/x_n : ... : x_{n-1}/x_n : 1]$
- Two projective lines in the same plane meet in at least one point



How on earth can parallel lines meet? Well..



- Projective geometry is the geometry of perspective
- Parallel lines meet at ∞ similarly as train tracks meet at the horizon
- Later this idea was formalized into projective space

Thank you for your attention!

I hope that was of some help.