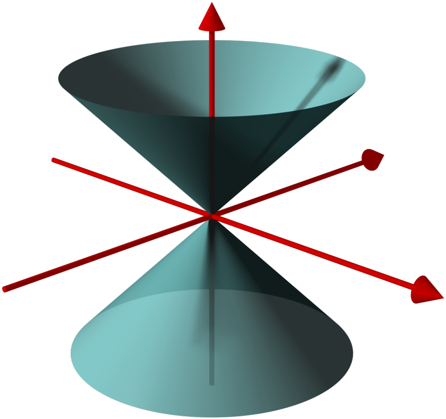
What are...cones?

Or: From affine to projective

Cone = double cone

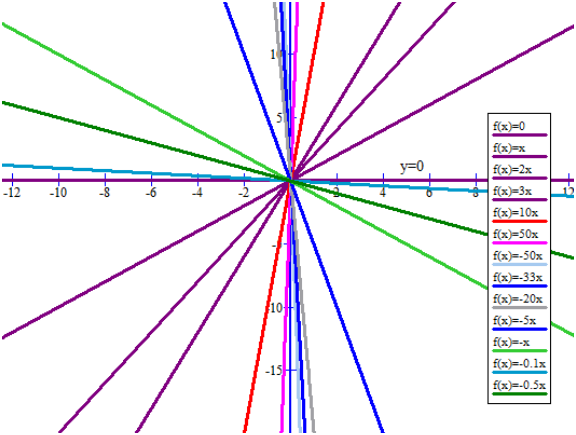
A cone :



, Not a cone :

- Cones play an important role in AG (e.g. via conic sections)
- Fun fact (later in this video) Every projective variety is a cone
- Careful In AG "cone" means "double cone"

Lines through the origin



- **Cone V** $=$ Affine variety with ($0 \in V$, and $\lambda v \in V$ for all $\lambda \in \mathbb{K}, v \in V$)
- **In other words** cones are unions of lines
- **Example** A line $x + y = 0$ is a cone

Homogeneous polynomials

$$h_0(X_1, X_2, \ldots, X_n) = 1,$$

$$h_1(X_1, X_2, \ldots, X_n) = \sum_{1 \le j \le n} X_j,$$

$$h_2(X_1, X_2, \ldots, X_n) = \sum_{1 \le j \le k \le n} X_j X_k,$$

$$h_3(X_1, X_2, \ldots, X_n) = \sum_{1 \le j \le k \le l \le n} X_j X_k X_l.$$

For $n = 1$:

$$h_1(X_1) = X_1 .$$

For $n = 2$:

$$h_1(X_1, X_2) = X_1 + X_2$$
$$h_2(X_1, X_2) = X_1^2 + X_1 X_2 + X_2^2.$$

For $n = 3$:

$$h_1(X_1, X_2, X_3) = X_1 + X_2 + X_3$$
$$h_2(X_1, X_2, X_3) = X_1^2 + X_2^2 + X_3^2 + X_1 X_2 + X_1 X_3 + X_2 X_3$$
$$h_3(X_1, X_2, X_3) = X_1^3 + X_2^3 + X_3^3 + X_1^2 X_2 + X_1^2 X_3 + X_2^2 X_1 + X_2^2 X_3 + X_3^2 X_1 + X_3^2 X_2 + X_1 X_2 X_3.$$

- Crucial example $V = \{v \in \mathbb{K}^n \mid f(v) = 0 \forall f \in P\}$ for P a set of homogeneous polynomials
- Here is why $f(\lambda v) = \lambda^{\deg f} f(v)$, so $f(v) = 0 \Leftrightarrow f(\lambda v) = 0$

For completeness: A formal statement

We have:

- **One direction** Every projective variety is a cone
- **The other direction** Every cone is a projective variety

- Formally:

$$\{\text{cones in } \mathbb{K}^{n+1}\} \xleftrightarrow{1:1} \{\text{projective varieties in } \mathbb{P}^n\}$$

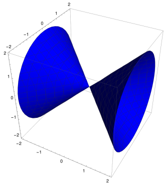
$$V \mapsto \text{projectivization } \pi(V) \text{ of } V$$

$$\text{cone } C(W) = \{0\} \cup \pi^{-1}(W) \text{ of } W \leftarrow\!\shortmid W$$

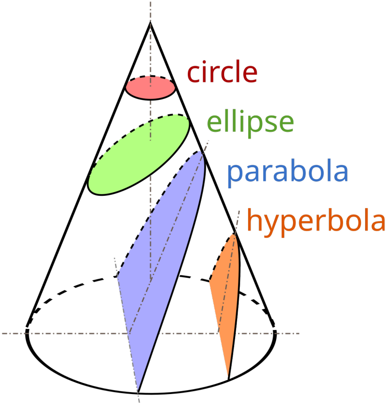
where $\pi\colon \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n, (x_0, ..., x_n) \mapsto (x_0 : ... : x_n)$

- From this point of view, projective varieties are easier than affine varieties

$x^2 - y^2 = 1 \rightsquigarrow x^2 - y^2 = z^2$:



The easier geometry



- Almost no affine variety corresponds to a projective one
- Projective geometry is easier than affine geometry, e.g. ellipse = parabola in projective land
- Careful "Easy" is subjective and context depending

Thank you for your attention!

I hope that was of some help.