What is...the projective Nullstellensatz?

Or: From projective to affine and back

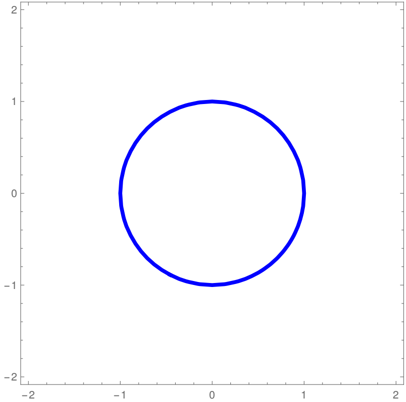
Projective is affine

Not quite the correct cone :

- Recall There is a bijection between cones in $\mathbb{K}^{n+1}$ and projective varieties in $\mathbb{P}^n$
- In other words every projective variety "is" an affine variety
- Conclusion "All" affine theorems should have projective counterparts

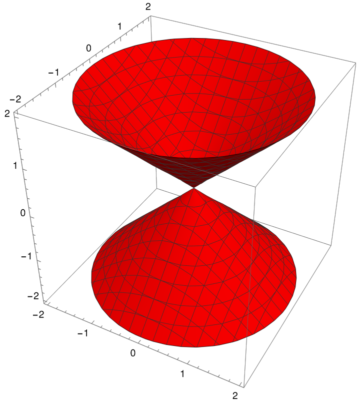
Affine is projective

$x^2 + y^2 - 1 = 0$



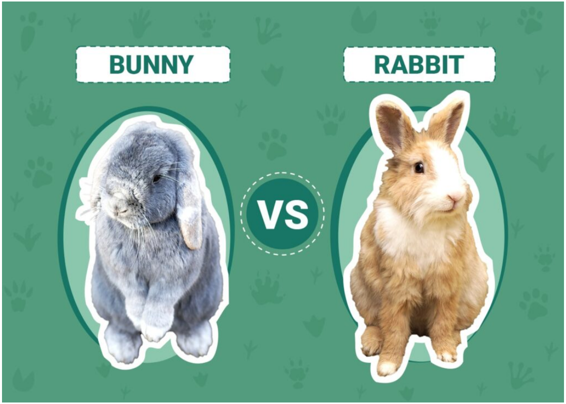
$\rightsquigarrow$

$x^2 + y^2 - z^2 = 0$



- Recall Making polynomials homogeneous goes from affine to projective varieties
- In other words every affine variety "is" a projective variety
- Conclusion "All" affine theorems should have projective counterparts

Affine ≈ projective



- There should be : a Zariski topology, regular functions, coordinate rings, ringed spaces etc. for projective varieties
- Mild adjustment Use homogeneous instead of general polynomials
- And yes this all works

For completeness: A formal statement

The projective Nullsellensatz holds for $\mathbb{K} = \bar{\mathbb{K}}$; this means:

- $V(I(X)) = X$ (for $X \subset \mathbb{P}^n$) An inverse
- $I(V(P)) = \sqrt{P}$ whenever $\sqrt{P} \neq I_0$ Almost an inverse

Same notation as before, but we only consider homogeneous polynomials

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- Here $I_0 = \langle x_0, \dots, x_n \rangle$ is the irrelevant ideal (the origin in $\mathbb{K}^{n+1}$ does not correspond to a projective variety)
- Compare Here is the affine Nullstellensatz:

We have Hilbert's Nullstellensatz:

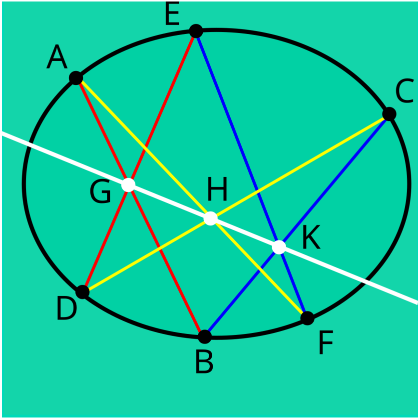
(i) $V(I(X)) = X$ An inverse

(ii) $I(V(P)) = \sqrt{P}$ Almost an inverse

Here our ground field is algebraically closed (e.g. $\mathbb{K} = \mathbb{C}$)

- $\sqrt{P} = \{f \in \mathbb{K}[x_1, \dots, x_n] \mid f^k \in P \text{ for some } k \in \mathbb{N}\}$ is the so-called radical

Is there a difference? Yes!



- Affine and projective are almost the same
- Both are special cases of general varieties
- Main difference Things that should intersect do in projective geometry

Thank you for your attention!

I hope that was of some help.