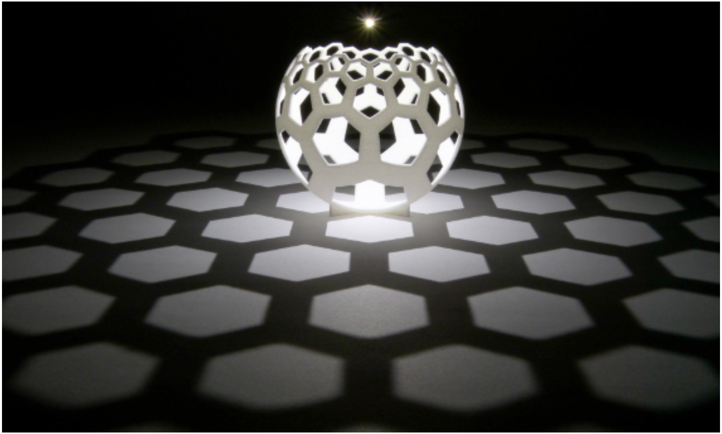
What is...the projective closure?

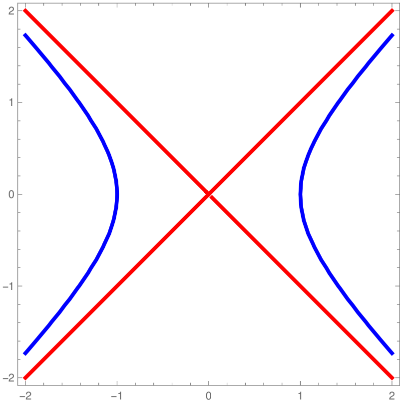
Or: Add points at infinity

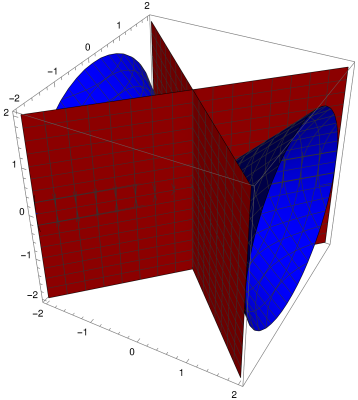
The Riemann sphere $\mathbb{P} = \mathbb{P}^1$



- Stereographic projection $=$ put a light source at the north pole ∞
- This defines a nice bijection between $\mathbb{C} \cup \{\infty\}$ and $\mathbb{P}$
- Note that $\mathbb{P}$ is compact (in the usual topology) – let's mimic this!

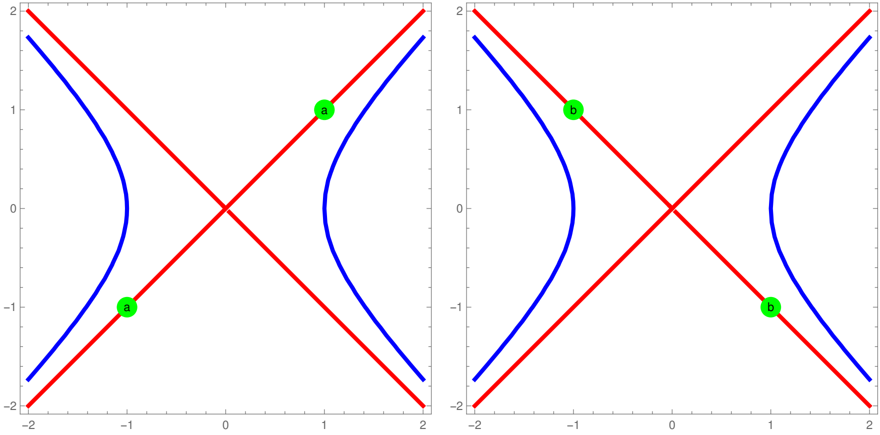
Make it homogeneous





- Above left The hyperbola $x^2 - y^2 - 1 = 0$ and $x^2 - y^2 = 0$
- Above right The homogeneous hyperbola $x^2 - y^2 - z^2 = 0$ and $x^2 - y^2 = 0$
- Observation $x^2 - y^2 - 1 = 0$ and $x^2 - y^2 = 0$ touch at infinity, and both are specializations of $x^2 - y^2 - z^2 = 0$

In projective land lines are points

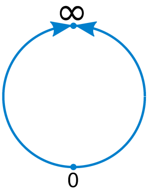


- Above The hyperbola $x^2 - y^2 - 1 = 0$ and $x^2 - y^2 = 0$
- Recall that $x = \lambda x$ in projective land, so the lines above are points in $\mathbb{P}^2$
- The points at ∞ are $a = (0 : 1 : 1)$, $b = (0 : 1 : -1)$

For completeness: A formal statement

The projective closure $\overline{V}$ of an affine variety V is:

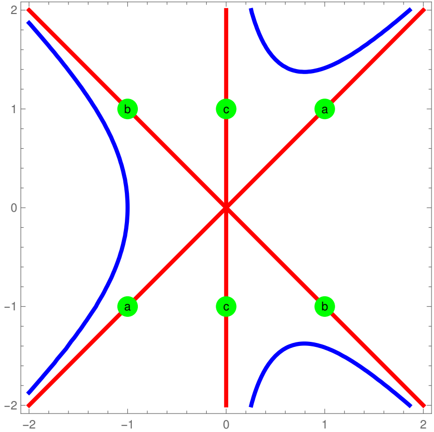
- If $V = V(f)$ for one polynomial f, make f homogeneous f^h and $\overline{V} = V(f^h)$
- The points at ∞ of $\overline{V}$ are the points with $x_0 = 0$



- There is a similar procedure for general V (skipped)

- Here $f = f(x_1, ..., x_n)$ and $f^h = (x_0, x_1, ..., x_n)$
- The projective closure is the smallest closed set in $\mathbb{P}^n$ containing V
- Since $\mathbb{P}^n$ is compact, so is $\overline{V}$ (in the usual topology)

A second example



- Above The cubic $x^3 - xy^2 + 1 = 0$ and $x^3 - xy^2 = 0$
- The projective closure is $x^3 - xy^2 + z^3 = 0$
- The points at ∞ are $a = (0 : 1 : 1)$, $b = (0 : 1 : -1)$, $c = (0 : 0 : 1)$

Thank you for your attention!

I hope that was of some help.