What is...the Segre embedding?

Or: Multiplying polynomials? Kind of!

Products are a bit nasty in AG

Products in AG

$$V = \mathbb{K} \Rightarrow \mathbb{K}[V] = \mathbb{K}[x]$$
$$W = \mathbb{K} \Rightarrow \mathbb{K}[W] = \mathbb{K}[y]$$
$$V \times W \cong \mathbb{K}^2 \Rightarrow \mathbb{K}[V \times W] = \mathbb{K}[x, y]$$

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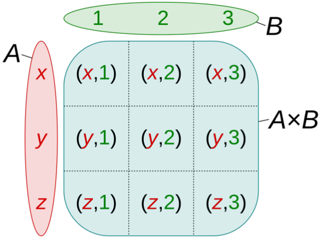


- For affine varieties $V \subset \mathbb{K}^m, W \subset \mathbb{K}^n$ the product $V \times W \subset \mathbb{K}^{m+n}$ is defined to have the Zariski topology not the product topology

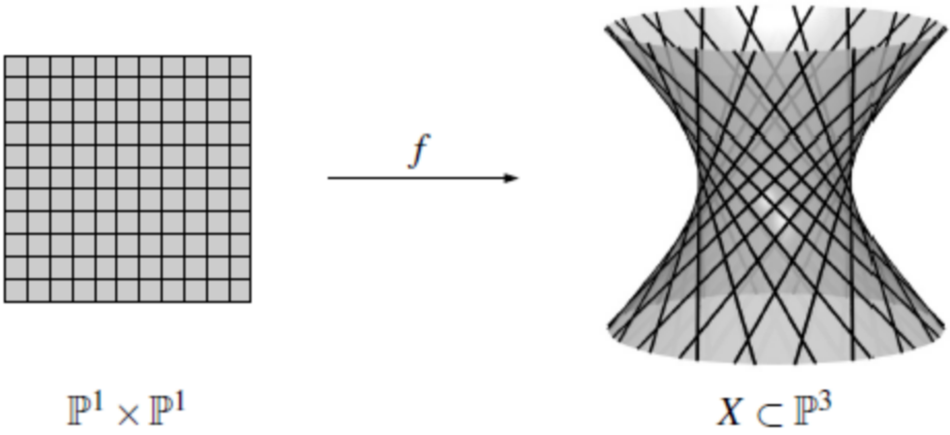
- Recall The product of affine varieties is an affine variety
- Mild catch The topology on the product is not the product topology
- In projective-land we can describe the product explicitly

Multiplying polynomials

	$2x^2$	$3x$	6
x	$2x^3$	$3x^2$	$6x$
-4	$-8x^2$	$-12x$	-24
	$2x^3-5x^2-6x-24$ ✓		

- **Above** The ‘correct’ way to multiply polynomials
- **Observation** Looks like a square
- **Idea** Maybe we should multiply $(x_0 : x_1)$ and $(y_0 : y_1)$ as polynomials

In projective land lines are points



- Task Give $\mathbb{P}^1 \times \mathbb{P}^1$ the structure of a projective variety
- Setup First copy of $\mathbb{P}^1$ consists of points $(x_0 : x_1)$, the second of points $(y_0 : y_1)$
- The map $((x_0 : x_1), (y_0 : y_1)) \to (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$ does a good job

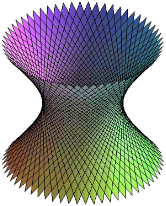
For completeness: A formal statement

The Segre embedding is:

$$f : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}, \big((x_0 : \dots : x_m), (y_0 : \dots : y_n)\big) \mapsto (\dots : x_i y_j : \dots)$$

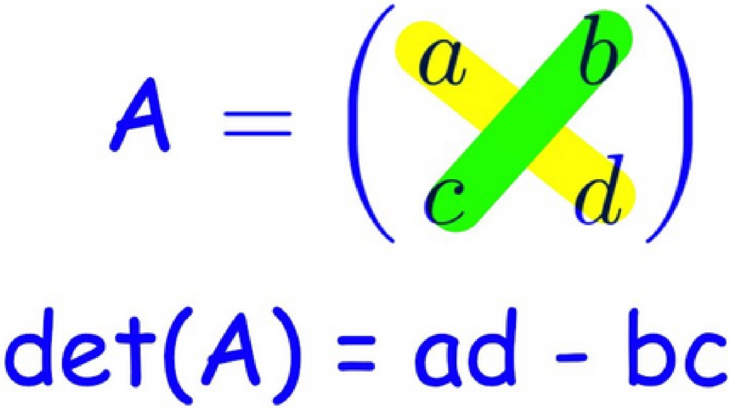
It identifies $\mathbb{P}^m \times \mathbb{P}^n$ with

$$V(z_{i,j}z_{k,l} - z_{i,l}z_{k,j} | 0 \leq i, k \leq m, 0 \leq j, l \leq n)$$



- In particular, $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety
- This implies that product of projective varieties is projective

The determinant



- Where does $V(z_{i,j}z_{k,l} - z_{i,l}z_{k,j} | 0 \leq i,k \leq m, 0 \leq j,l \leq n)$ come from?
- Example For $m = n = 1$ this is $z_{0,0}z_{1,1} - z_{0,1}z_{1,0}$
- This is the determinant of $\begin{pmatrix} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{pmatrix}$

Thank you for your attention!

I hope that was of some help.