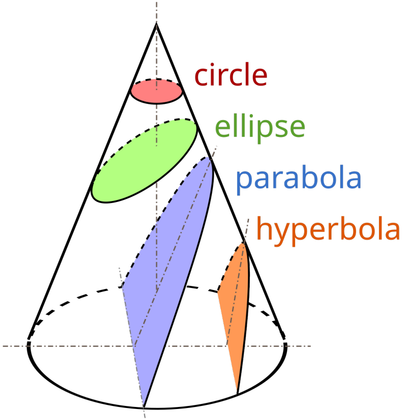
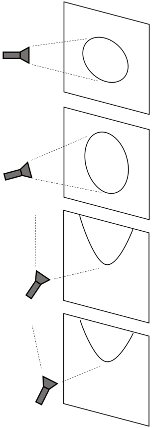
What is...the Veronese embedding?

Or: Symmetric powers

Conic sections again

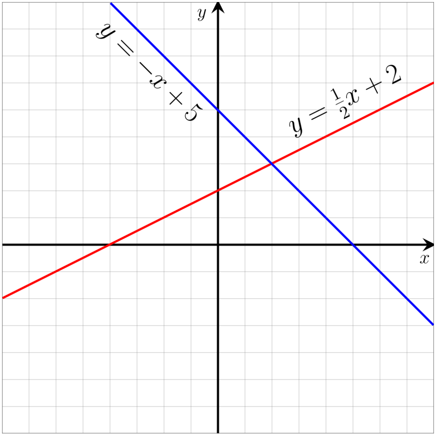




- Reminder Conic sections are circles, ellipse, etc.
- Generic equation of degree two

$$A \cdot x^2 + B \cdot xy + C \cdot y^2 + D \cdot xz + E \cdot yz + F \cdot z^2 = 0$$

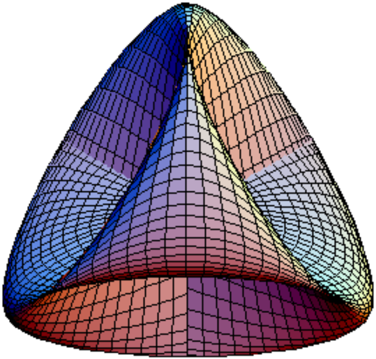
Points on conics



- Turn it around!
- Fix a point $(x : y : z)$
- The condition that a conic contains $(x : y : z)$ is linear in (A, B, C, D, E, F)

Veronese surface

3d linear projection of the image :



- Veronese embedding sends points $(x : y : z)$ to "coefficients" $(x^2 : xy : y^2 : xz : yz : z^2)$; Reordered we get

$$\mathbb{P}^2 \to \mathbb{P}^5, (x : y : z) \mapsto (x^2 : y^2 : z^2 : xy : xz : yz)$$

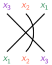
For completeness: A formal statement

The dth Veronese embedding is

$$\mathbb{P}^n \to \mathrm{Sym}^d(\mathbb{P}^n), [v] \mapsto [v^d]$$

- Reminder on symmetric spaces

Take the symmetric group and act on $\mathbb{R}[X_1, X_2, X_3]$ by permuting the variables.
Symmetric polynomials? These are fixed by permutation, *e.g.*



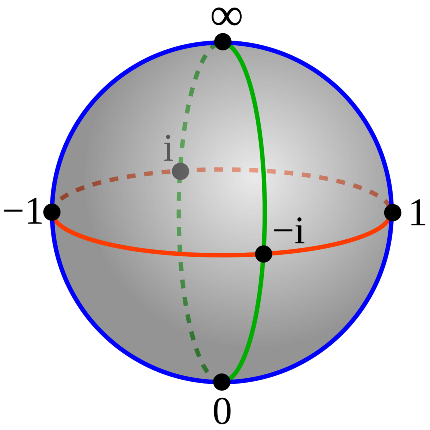
$$X_1X_2 + X_1X_3 + X_2X_3 \longrightarrow X_3X_2 + X_3X_1 + X_2X_1 = X_1X_2 + X_1X_3 + X_3X_2$$

degree	polynomials	symmetric basis
0	1	1
1	$X_1 + X_2 + X_3$	X_1, X_2, X_3
2	$X_1X_2 + X_1X_3 + X_2X_3$, $X_1^2 + X_2^2 + X_3^2$	$X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3$

These are symmetric polynomials are also called symmetric tensors and they live inside the symmetric algebra (but honestly, so they are not spanning it).

- Example $n = 2, d = 2$ was on the previous slide

A crucial corollary



- Corollary A projective variety minus a zero set of a homogeneous non-constant regular function is affine
- The Proof (sketch) If the regular function is linear this is easy; otherwise apply the dth Veronese embedding where d is the degree

Thank you for your attention!

I hope that was of some help.