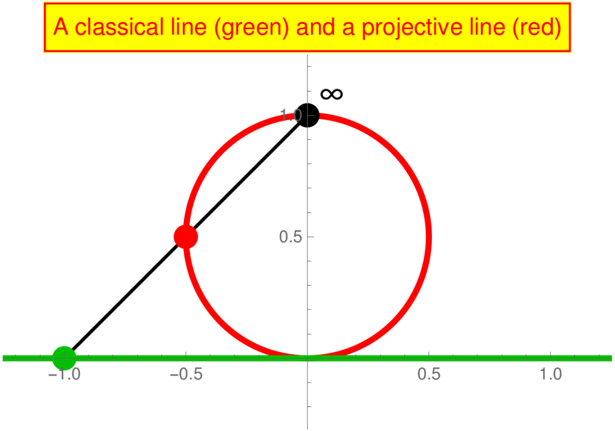
What are...Grassmannians?

Or: A space in a space

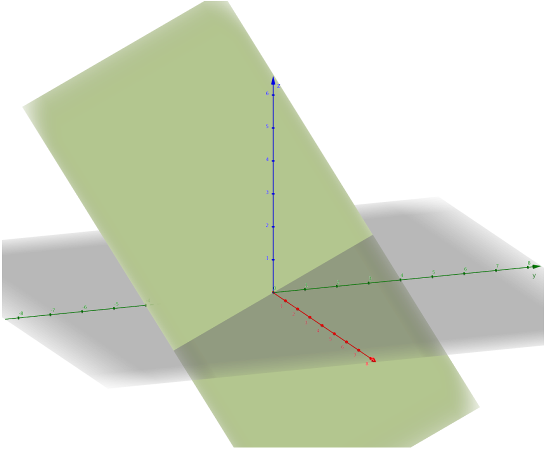
Projective space = lines



- Reminder Projective space $\mathbb{P}^n$ is equivalent to lines in $\mathbb{K}^{n+1}$
- Alternatively $\mathbb{P}^n$ over $\mathbb{R}$ is $S^{n+1}/antipodes$
- $\mathbb{P}^n$ is one of the most important spaces in AG and also topology

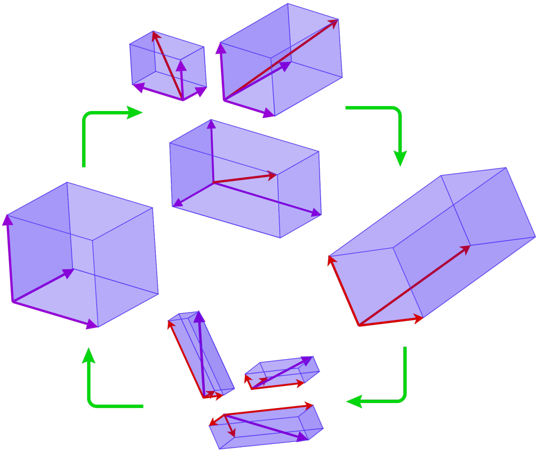
Grassmannian = higher lines

An element of $G(2,3)$:



- Grassmannian $G(k,n)$ is the set of k-planes in $\mathbb{K}^n$ (here $k \in \{0,\dots,n\}$)
- Boring examples $G(0,n)$ and $G(n,n)$ are points
- Good example $G(1,n) = \mathbb{P}^{n-1}$, so we generalize projective space

Matrices!



- After picking a basis, $G(k,n) =$ the elements are n-by-k matrices of rank k
- $M = N$ see in $G(k,n) \Leftrightarrow$ $M = Ng$ (base change) for $g \in \mathrm{GL}_k(\mathbb{K})$
- Example For $k = 1$ this matches homogeneous coordinates

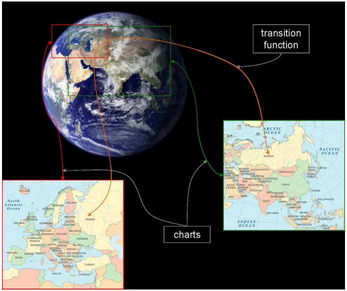
For completeness: A formal statement

The kth Grassmannian, say for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is

(i) An affine variety

(ii) A projective variety Great!

(iii) A differentiable (real or complex) manifold



- Next video: $G(k, n)$ is a projective variety
- The dimension of $G(k, n)$ is $k(n-k)$ (next slide)

The column echelon form

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{1,1} & \cdots & \cdots & a_{1,k} \\ \vdots & & & \vdots \\ a_{n-k,1} & \cdots & \cdots & a_{n-k,k} \end{bmatrix}$$

- Reduced column echelon form = defined above by example
- Elements of $G(k,n)$ correspond 1:1 to column echelon matrices as above
- The $k(n-k)$ free coordinates are the variables

Thank you for your attention!

I hope that was of some help.