What are...Plücker coordinates?

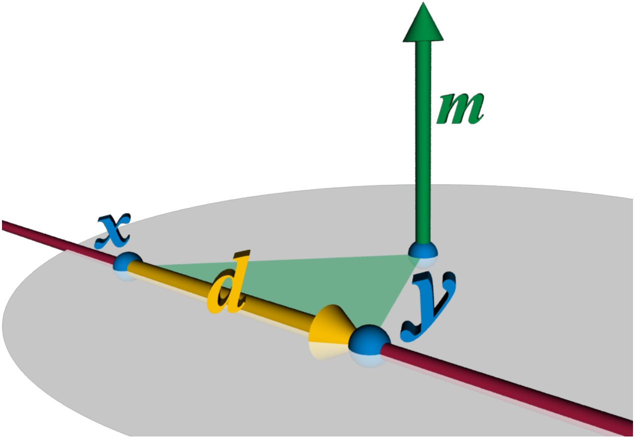
Or: Exterior powers

The Grassmannian

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{1,1} & \cdots & \cdots & a_{1,k} \\ \vdots & & & \vdots \\ a_{n-k,1} & \cdots & \cdots & a_{n-k,k} \end{bmatrix}$$

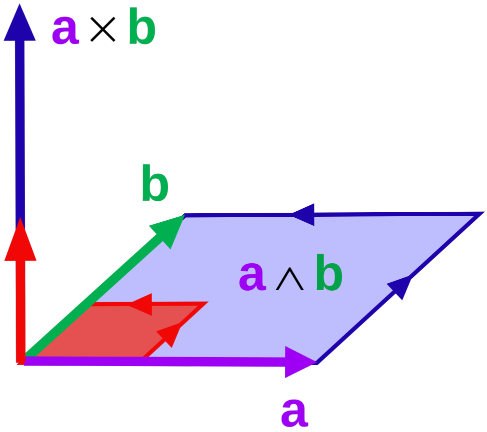
- **Reminder** Grassmannian $G(k,n) = k$-planes in n-space
- **Task** Show that $G(k,n)$ is a variety
- **Not trivial** What polynomial vanishing set is $G(k,n)$?

Points with momentum



- $G(2,4)$ = planes in four-space
- Convention Lets assume this is a particle with momentum
- Observation This is determined by $d = y - x$ and $m = x$ cross y

Exterior space



- $G(2,3)$ = planes in three-space
- **Observation** A plane is determined by two vectors a, b
- **Better** A plane is determined by the parallelogram $a \wedge b$

For completeness: A formal statement

The kth Plücker embedding is

$$G(k,n) \to \mathrm{Ext}^k(\mathbb{P}^n), \mathrm{span}(v_1, ..., v_k) \mapsto v_1 \wedge ... \wedge v_k$$

- Reminder on exterior spaces

The polynomial algebra

$$\mathbb{R}[X_1, X_2, X_3] = \mathbb{R}\langle X_1, X_2, X_3\rangle/(X_iX_j = X_jX_i)$$

Variables commute

The exterior algebra

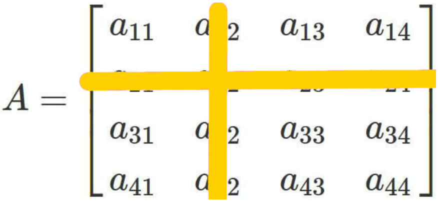
$$\mathrm{Ext}(X_1, X_2, X_3) = \mathbb{R}\langle X_1, X_2, X_3\rangle/(X_iX_j = -X_jX_i)$$

Variables anticommute

- Examples $n = 3, k = 2$ was on the previous slide, on the slide before it was $n = 4, k = 2$ (projective lines in real projective space, correspond to 2d subspaces of a 4d vector space)

Many minors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$



- $G(2,4)$ = 'volumes' in four space
- The Plücker embeding realizes this in $\mathbb{P}^5$ (since $\binom{4}{2} = 6$)
- It is projective! In this interpretation $G(2,4)$ = zero sets of the sixteen 3-by-3 minors of the 4-by-4 matrix $\mathbb{K}^4 \to \wedge^3\mathbb{K}^4, v \mapsto v \wedge (a \wedge b)$

Thank you for your attention!

I hope that was of some help.