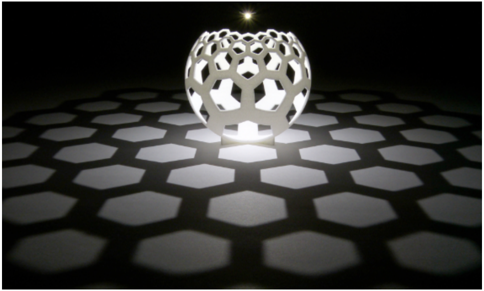
What are...Grassmannians, take 2?

Or: A crucial example

Matrices and $G(k,n)$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{1,1} & \cdots & \cdots & a_{1,k} \\ \vdots & & & \vdots \\ a_{n-k,1} & \cdots & \cdots & a_{n-k,k} \end{bmatrix}$$



- Reminder $G(k,n)$ can be identified with full rank k-by-n matrices
- The equivalence is given by elementary column operations
- Example $G(1,2)=\mathbb{P}^1$: $(1\ \ *) \leftrightsquigarrow \mathbb{K}$ and $(0\ \ 1) \leftrightsquigarrow \infty$

Column reduction – details

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{5}{3} & -\frac{4}{3} & 1 \end{pmatrix}$$

- Example The element of $G(3,6)$ above in normal form (right)
- Going from left to right is Gaussian elimination
- Alternatively Use the transpose picture and row reduction (slightly nicer)

Schubert cells

$[0,0]$	$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$	4
$[1,0]$	$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$	3
$[1,1]$	$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$	2
$[2,0]$	$\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	2
$[2,1]$	$\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	1
$[2,2]$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	0

- $G(2,4)$ This is indexed by the partitions that fit into a 2-by-$(4-2)$ box
- Dimension $=$ number of free parameters
- Partitions First number: # left zeros in the second row -1, second number $=$ # left zeros in the first row

For completeness: A formal statement

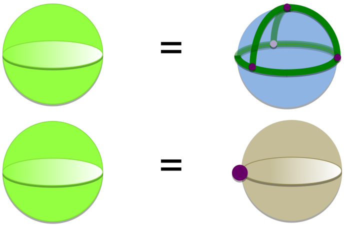
The Schubert cells S_λ give $G(k, n)$ the structure of a cell complex with:

(i) The partitions λ one needs to consider are those that fit into a $k(n-k)$ box

(ii) A partition with $k(n-k) - j$ boxes is a cell of dimension j

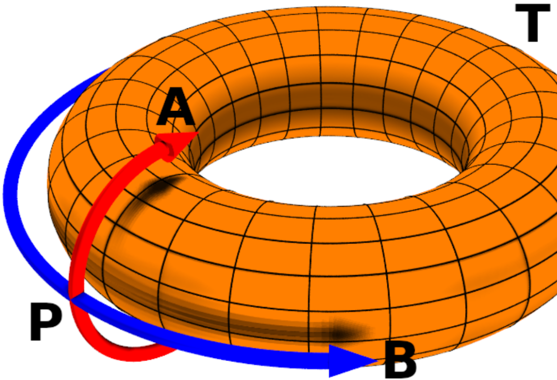
(iii) There is at least one cell for each $j \in \{0, \dots, k(n-k)\}$; there are $\binom{n}{k}$ cells

(iv) The (closure of the) S_λ (mimicking the previous slide) are subvarieties



- For this video I take complex coefficients
- The attaching maps are a bit nasty and skipped

The cohomology ring



$H_0(T) \cong \mathbb{Z} \leftrightsquigarrow [P], H_1(T) \cong \mathbb{Z}^2 \leftrightsquigarrow [A], [B], H_2(T) \cong \mathbb{Z} \leftrightsquigarrow [T]$

- Cohomology ring = a multiplication structure on the cells
- Cells of $G(k, n)$ correspond 1:1 to Schur polynomials
- The multiplication of the cohomology ring is the one of Schur polynomials

Thank you for your attention!

I hope that was of some help.