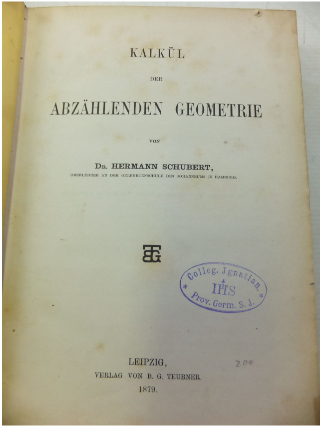
What is...Schubert calculus?

Or: Intersecting spaces

Schubert's book



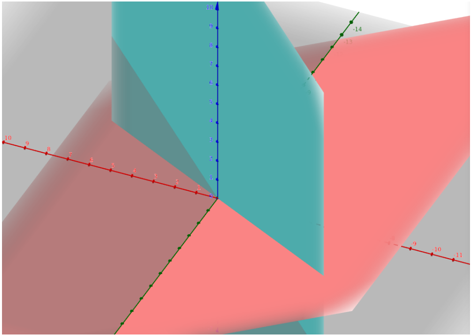
- Schubert calculus = a method to solve various geometric counting problems (e.g. "How often does XYZ intersect ABC?")
- These were published by Schubert in 1879
- Problem The original ideas were 'Euler-style': brilliant, but not rigorous

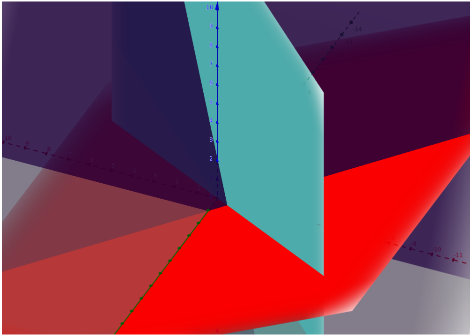
Hilbert's question

13th	Solve 7th-degree equation using algebraic (variant: continuous) functions of two parameters.	Unresolved. The continuous variant of this problem was solved by Vladimir Arnold in 1957 based on work by Andrei Kolmogorov (see Kolmogorov–Arnold representation theorem), but the algebraic variant is unresolved.[k]	—
14th	Is the ring of invariants of an algebraic group acting on a polynomial ring always finitely generated?	Resolved. Result: No, a counterexample was constructed by Masayoshi Nagata.	1959
15th	Rigorous foundation of Schubert's enumerative calculus.	Partially resolved.[23]	—
16th	Describe relative positions of ovals originating from a real algebraic curve and as limit cycles of a polynomial vector field on the plane.	Unresolved, even for algebraic curves of degree 8.	—

- Hilbert's 23 problems are among the most influential questions in mathematics
- These were asked by Hilbert in 1900
- Among them: Make Schubert's ideas rigorous

A first solution





- Easy but crucial Generically, two planes in three space intersect in a line, three in a point and four have no common intersection
- Cohomology ring $H^*(\mathbb{P}^3) \cong \mathbb{Z}[x]/x^4$, and:
  - $x \leftrightsquigarrow$ plane
  - $x^2 \leftrightsquigarrow$ plane $\cap$ plane $=$ line
  - $x^3 \leftrightsquigarrow$ plane $\cap$ plane $\cap$ plane $=$ point
  - $x^4 \leftrightsquigarrow$ plane $\cap$ plane $\cap$ plane $\cap$ plane $=$ dead

For completeness: A formal statement

The cohomology ring of $G(k, n)$ is:

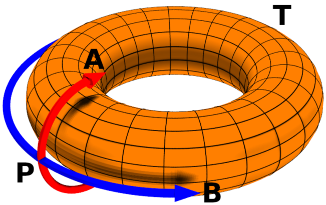
(i) It is a quotient of symmetric polynomials in $x_1, ..., x_k$

(ii) The quotient is given by annihilating the complete symmetric polynomial with $n - k + 1$ nodes

(iii) It has a basis by Schur polynomials s_λ that fit into a $k(n-k)$ box

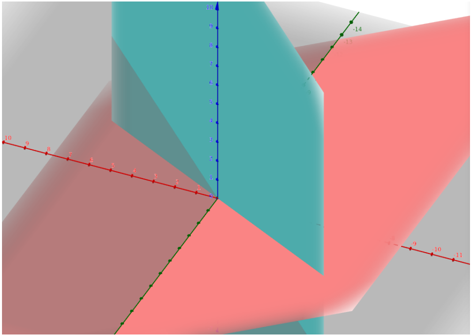
(iv) The s_λ have explicit structure constants (Littlewood–Richardson coefficients)

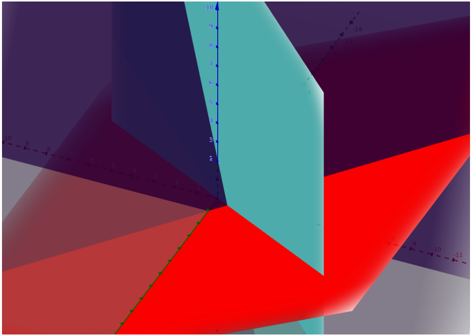
- This explicitly describes Schubert's calculus
- Why? Because cohomology ring = intersections, and s_λ = subspaces



$H_0(T) \cong \mathbb{Z} \leftrightsquigarrow [P], H_1(T) \cong \mathbb{Z}^2 \leftrightsquigarrow [A], [B], H_2(T) \cong \mathbb{Z} \leftrightsquigarrow [T]$

Example





- Recall $H^*(\mathbb{P}^3) \cong \mathbb{Z}[x]/x^4$ and $\mathbb{P}^3 \cong G(1,3)$
- Indexing set $\emptyset \leftrightsquigarrow 1$, $\square \leftrightsquigarrow x$, $\square\square \leftrightsquigarrow x^2$, $\square\square\square \leftrightsquigarrow x^3$, $\square\square\square\square \leftrightsquigarrow$ kill,
- In general, $H^*(\mathbb{P}^n) \cong \mathbb{Z}[x]/x^{n+1}$ and $\mathbb{P}^n \cong G(1,n)$; $x \leftrightsquigarrow (n-1)$ hyperplane
- The picture is dualized and we take a (hyper)plane and not a line

Thank you for your attention!

I hope that was of some help.