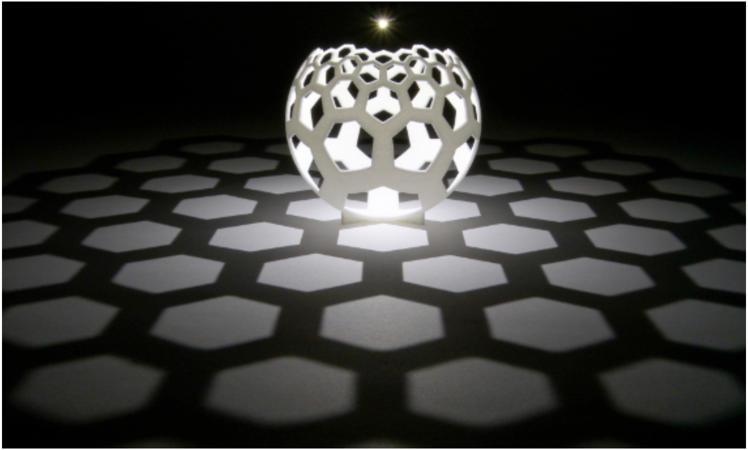
What are...birational maps?

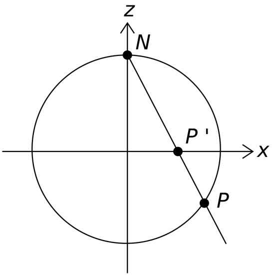
Or: Affine is projective

Stereographic projection (SP)



- Affine space = the plane
- Projective space = the sphere
- Question They are essentially the same – how to make this rigorous?

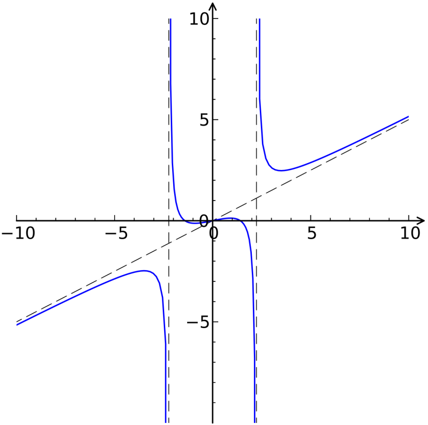
Stereographic projection – coordinates (shifted)



$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$
$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right).$$

- SP identifies affine and projective space – except for the north pole
- (X, Y) = coordinates of the plane
- (x, y, z) = coordinates of the sphere

Not a polynomial!



- Observation 1 SP is not given by a polynomial but rather by a rational function
- Observation 2 SP is not defined on the whole sphere but only on an open subset
- Idea Use this to define our equivalence

For completeness: A formal statement

We define an equivalence relation on varieties by:

(i) Rational maps $V \to W$ are morphism from a nonempty open subset to W

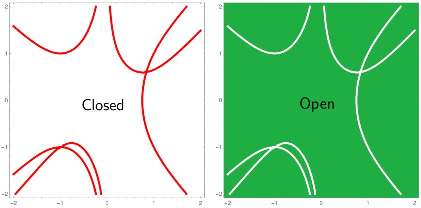
(ii) A birational map $V \to W$ is a rational map with a rational inverse

(iii) $V \cong_b W$ (birationally equivalent) if there exists a birational map $V \to W$

- Example Affine space is birationally equivalent to projective space
- Recall Open sets are large, so ‘from a nonempty open subset’ is not a huge restriction

The Zariski topology is a bit obscure, e.g.:

- Closed sets are very small, open sets are very large



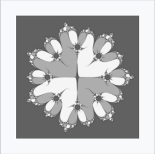
- One cannot separate points

Birational is difficult

$$\frac{1}{az^5 + z^3 + bz}$$

$$\frac{1}{z^3 + z(-3 - 3i)}$$

$$\frac{z^2 - 0.2 + 0.7i}{z^2 + 0.917}$$



$$\frac{z^2}{z^9 - z + 0.025}$$

- Careful Rational maps are much more intricate than polynomials
- Example Julia sets arise as iterations of rational functions
- And indeed, birational equivalence is difficult

Thank you for your attention!

I hope that was of some help.