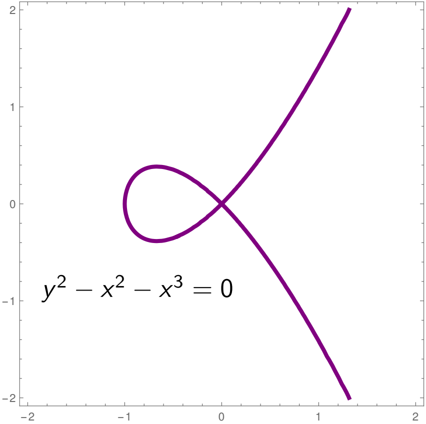
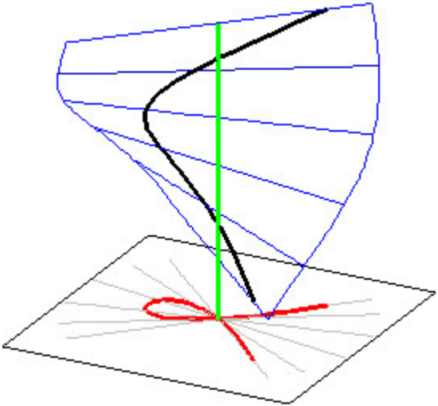
What is a...blow up, take 2

Or: Playing with a time coordinate

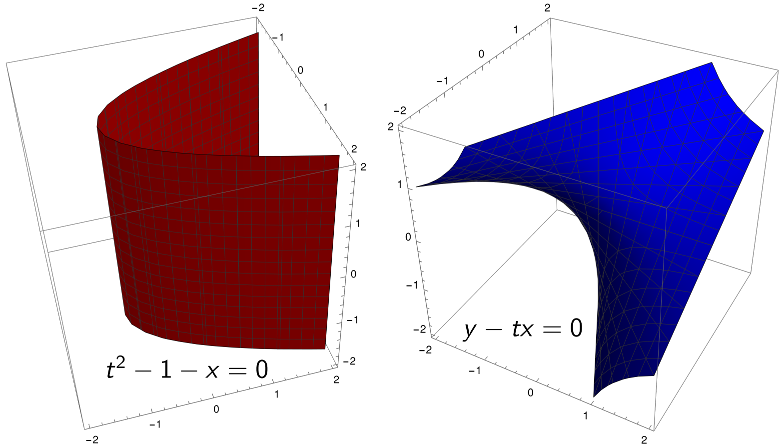
Running example





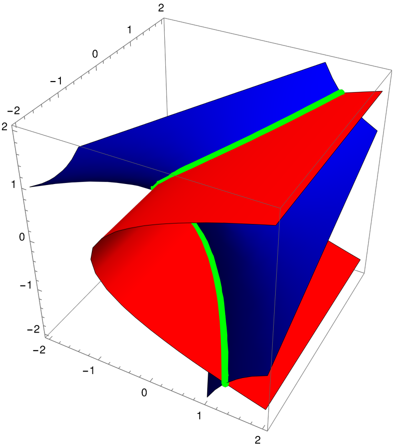
- Example Take the affine variety $V(y^2 - x^2 - x^3)$
- Recall Add a time direction t to avoid the singular point at the origin
- Question How do we describe the right side from $y^2 - x^2 - x^3 = 0$?

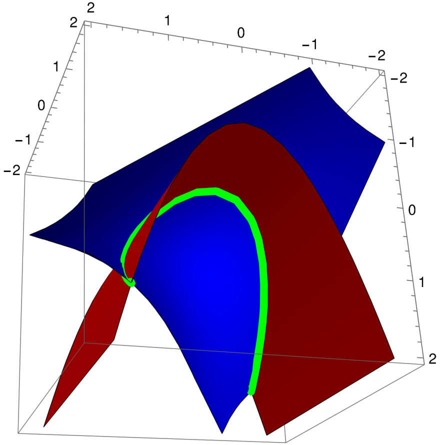
Two equations



- Idea Birationally, using $t = y/x$ doesn't change the variety
- We get two equations $t^2 - 1 - x = 0$ and $y - tx = 0$
- This takes place in $\mathbb{K}^2 \times \mathbb{P}^1$ with $\mathbb{P}^1$ identified as 'time'

Intersection





- **Above** The intersection of $t^2 - 1 - x = 0$ and $y - tx = 0$
- **Looks good!**
- **Summary** $\mathbb{K}^2 \setminus V(x,y) \to \mathbb{P}^1, (x,y) \mapsto (y/x : 1)$ does the job

For completeness: A formal statement

For some affine variety $V \subset \mathbb{K}^n$ fix:

(i) Polynomials $f_1, \ldots, f_k \in \mathbb{K}[V]$

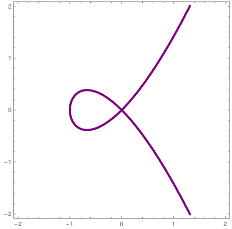
(ii) $U = V \setminus V(f_1, \ldots, f_k)$

(iii) The evaluation $f : U \to \mathbb{P}^{n-1}, x \mapsto (f_1(x) : \ldots : f_r(x))$

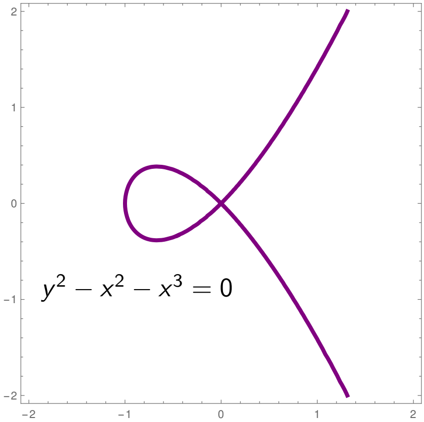
(iv) The graph $\Gamma_f = \{(x, f(x))\} \subset U \times \mathbb{P}^{n-1}$

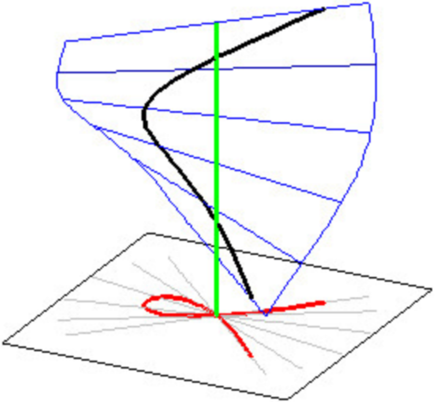
The closure of Γ_f, denote $\tilde{V}$ is the blow-up of V at $f_1, \ldots, f_k$

- The is a map $\pi : \tilde{V} \to V$, and $\pi^{-1}(V)$ is the exceptional set
- Next slide How does our example fit into this for the blow up ?



Zoom into the example





- What we have seen $\tilde{V}$ is $\mathbb{K}^2$ blown up at the coordinate functions x, y
- What we have seen is the transformation of $V(y^2 - x^2 - x^3)$ at x, y, that is, $\tilde{V}(y^2 - x^2 - x^3)$ realized within $\tilde{V}$
- Fact $\tilde{V}(y^2 - x^2 - x^3)$ is the closure of $V(y^2 - x^2 - x^3) \cap U$ in $\tilde{V}$

Thank you for your attention!

I hope that was of some help.