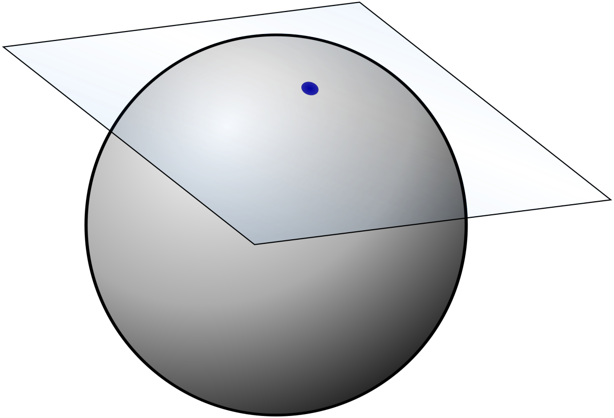
What is a...blow up, take 3?

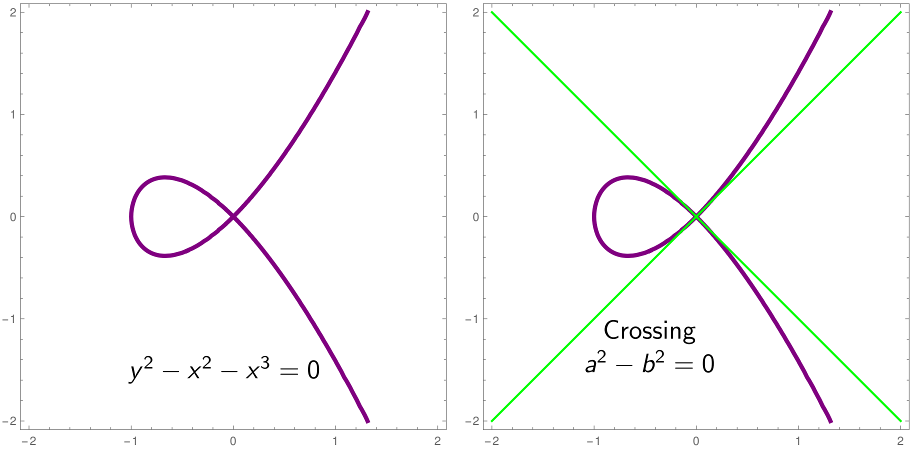
Or: Tangent cones

Tangent spaces



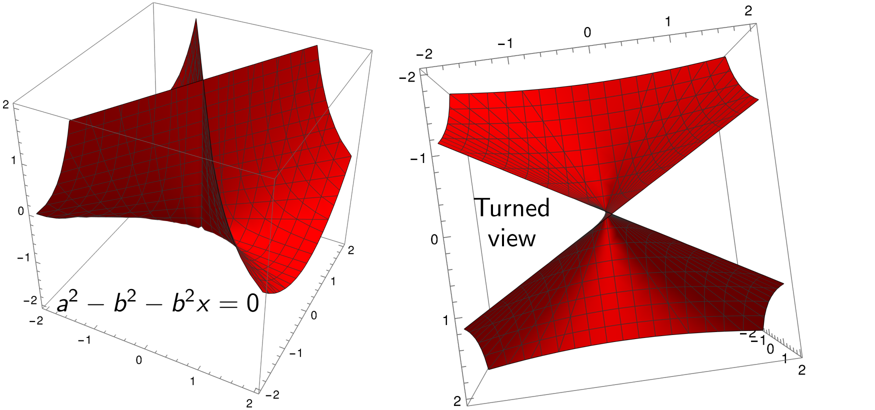
- Tangents (= lowest order approximations) are everywhere in math
- Example above The tangent plane for $x^2 + y^2 + z^2 - 1 = 0$
- Question What are tangent spaces for singularities?

Tangent spaces?



- The curve above has a singularity at $(0,0)$
- Indeed, the partial derivatives vanish at the point $(0,0)$
- Observation The 'crossing' is the limit $(x,y) \to (0,0)$ of two tangents

Adding time again



- Change of variables $y = at$, $x = bt$

  $$(y^2 - x^2 - x^3 = 0) \Rightarrow (a^2t^2 - b^2t^2 - b^2t^2x = 0) \Rightarrow (a^2 - b^2 - b^2x = 0)$$

- At our singularity we have $x = 0$, so $a^2 - b^2 = 0$

- Why does this work?

For completeness: A formal statement

For some affine variety $V \subset \mathbb{K}^n$ and some point $v \in V$:

(i) Consider the blow-up $\tilde{V}$ of V at $x - v$

(ii) We have $\pi\colon \tilde{V} \to V$, and $\pi^{-1}(\{v\})$ is the exceptional set (a projective variety)

(iii) The cone over $\pi^{-1}(\{v\})$ is called the tangent cone

This replaces the tangent for singular spaces

- Cone = "union of lines" (see video "What are...cones?")

- Formally:

$$\{\text{cones in } \mathbb{K}^{n+1}\} \overset{1:1}{\longleftrightarrow} \{\text{projective varieties in } \mathbb{P}^n\}$$

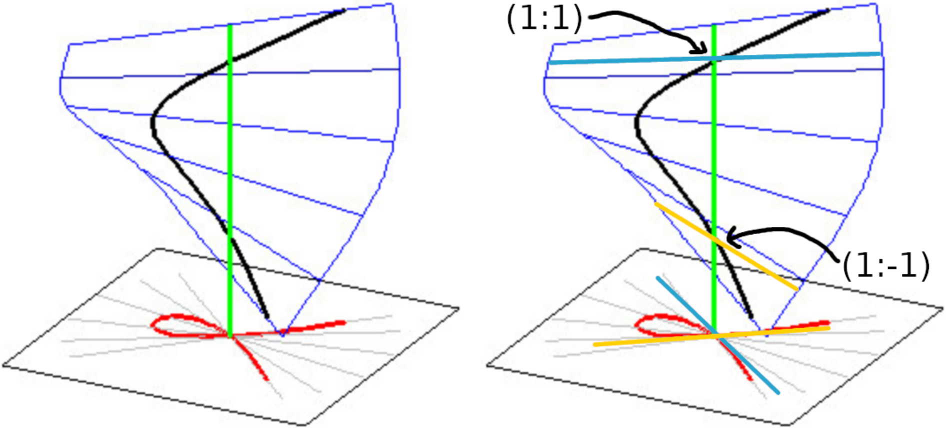
$$V \mapsto \text{projectivization } \pi(V) \text{ of } V$$

$$\text{cone } C(W) = \{0\} \cup \pi^{-1}(W) \text{ of } W \leftarrow\!\shortmid W$$

where $\pi\colon \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n, (x_0, \dots, x_n) \mapsto (x_0 : \dots : x_n)$

- Next slide How does our example fit into this?

Zoom into the example



- Green horizontal line $= \pi^{-1}(\{0\})$ in $\tilde{\mathbb{K}^2}$; $\pi^{-1}(\{0\})$ in $\tilde{V} = (1 : \pm 1)$
- Cone $=$ union of lines in gray for $\tilde{\mathbb{K}^2}$; restrict to the ones touching $(1 : \pm 1)$
- Our example Lowest order approximation (equivalent to calculating tangent cones)

Thank you for your attention!

I hope that was of some help.