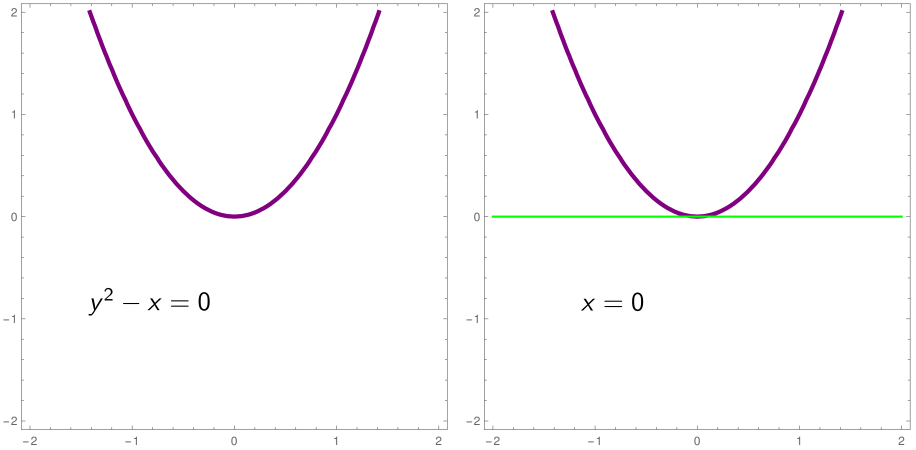
What are...tangent cones?

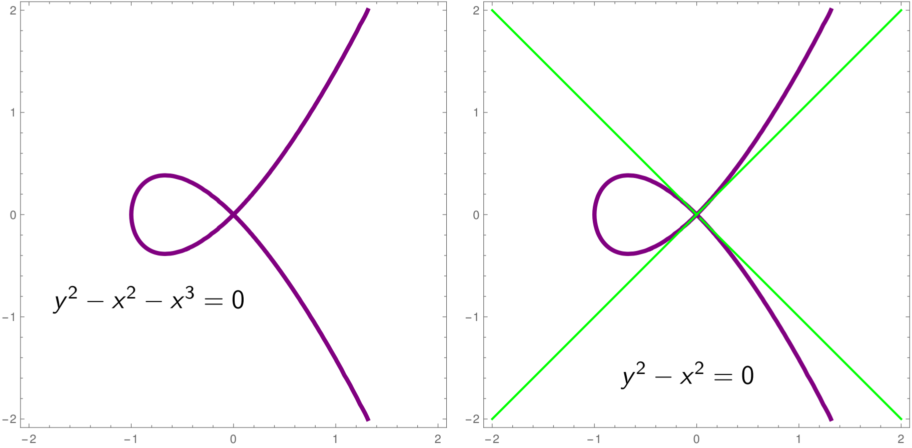
Or: How to compute them

Example 1



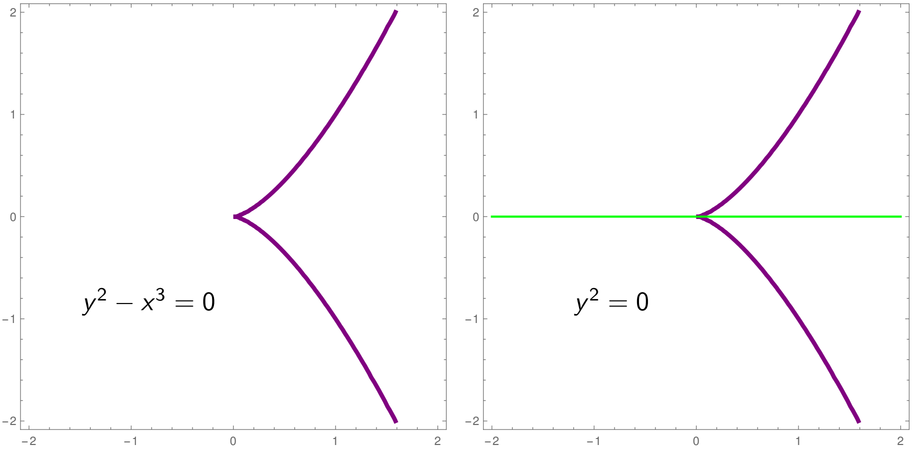
- Example Take the affine variety $y^2 - x = 0$
- This is smooth at the origin
- Tangent cone (at $(0, 0)$) is $V(x = 0)$

Example 2



- Example Take the affine variety $y^2 - x^2 - x^3 = 0$
- This has a double point at the origin
- Tangent cone (at $(0, 0)$) is $V(y^2 - x^2 = 0)$

Example 3



- Example Take the affine variety $y^2 - x^3 = 0$
- This has a cusp point at the origin
- Tangent cone (at $(0,0)$) is $V(y^2 = 0)$

For completeness: A formal statement

The tangent cone can be calculated as follows:

(i) The initial term f^i of f is the sum of all monomials of smallest degree

(ii) Say $V = V(f_1, ..., f_k)$

(iii) Then the tangent cone at the origin is $C(V) = V(f^i | f \in \langle f_1, ..., f_k \rangle)$

(iv) For other points shift the coordinates to the origin

Thus, tangent cone $=$ forget higher degree terms

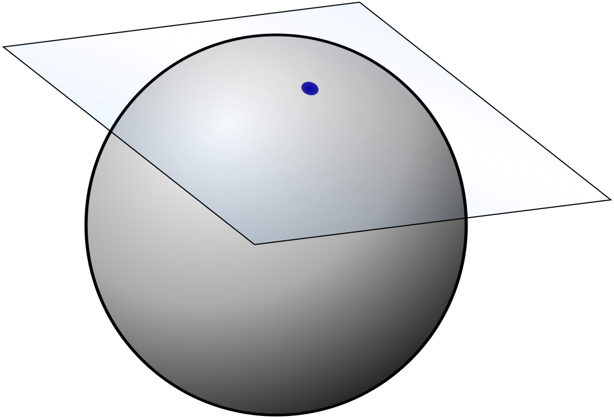
- Example For $f = y^2 - x$ we have $f^i = x$, matching the first example
- Recall A homogeneous polynomial $\leftrightsquigarrow$ projective variety $\leftrightsquigarrow$ cone

- Formally:

$$\{\text{cones in } \mathbb{K}^{n+1}\} \xleftrightarrow{1:1} \{\text{projective varieties in } \mathbb{P}^n\}$$
$$V \mapsto \text{projectivization } \pi(V) \text{ of } V$$
$$\text{cone } C(W) = \{0\} \cup \pi^{-1}(W) \text{ of } W \leftarrow W$$

where $\pi\colon \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n, (x_0, ..., x_n) \mapsto (x_0 : ... : x_n)$

The dimension is good



- Recall For a smooth manifold M, the dimension of the tangent space is $\dim M$
- Bad The tangent cone $C(V)$ is often not linear
- Good The dimension of $C(V)$ is the expected one, namely $\dim V$

Thank you for your attention!

I hope that was of some help.