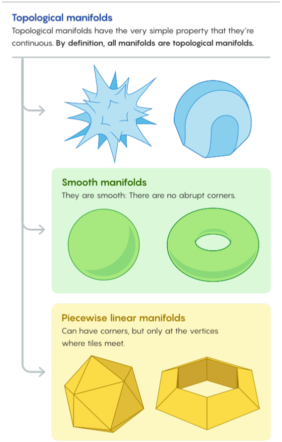
What are...smooth varieties?

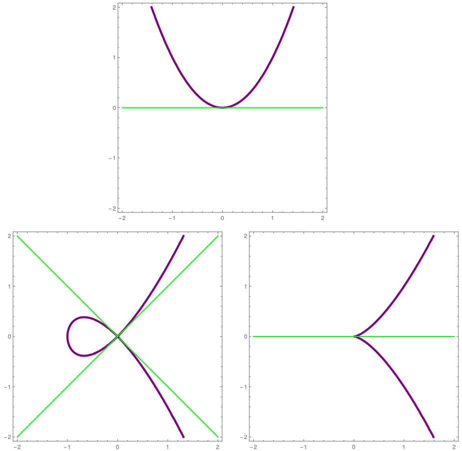
Or: Not smooth is rare

Smooth manifolds



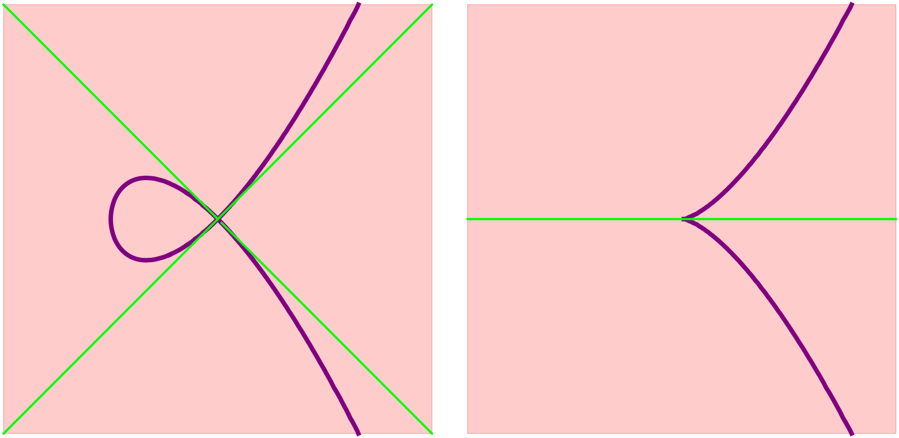
- Recall A function $f\colon \mathbb{C} \to \mathbb{C}$ is differentiable $\Leftrightarrow$ it is analytic/smooth
- Differentiable = a tangent can be defined; Smooth = no corners
- Ansatz Define what it means to be smooth as a variety using a ‘tangent’

Two equations



- Top ("no corners") The tangent cone is a tangent
- Bottom ("corners") The tangent cone is not a tangent
- Idea Define a tangent as a linear tangent cone

Tangent = linear approximation



- Above The two equations are $y^2 - x^2 - x^3 = 0$ and $y^2 - x^3 = 0$
- There is no linear term so the tangent should be $V(0) =$ all of space
- Observation The tangent cone is always inside of the tangent

For completeness: A formal statement

The tangent (at the origin) is define as follows:

(i) The linear term f^l of f is the sum of all monomials of degree one

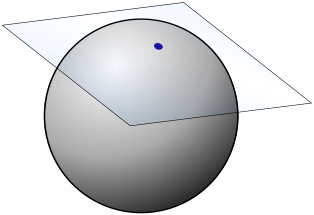
(ii) Say $V = V(f_1, \ldots, f_k)$

(iii) Then the tangent at the origin is $T(V) = V(f^l | f \in \langle f_1, \ldots, f_k \rangle) \supset C(V)$

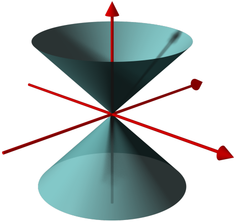
(iv) For other points shift the coordinates to the origin

Thus, tangent = forget nonlinear terms

- Definition Smooth $\rightsquigarrow$ $C(V) = T(V)$ (tangent cone equals tangent)
- Next video Why does this agree with the usual definition of smooth?



Singularities are rare



- Singularities, in life and mathematics, are “point-events that change everything”
- Indeed, we have: the set of smooth points is open
- Recall Open sets are large (e.g. for irreducible varieties they are dense)

Thank you for your attention!

I hope that was of some help.