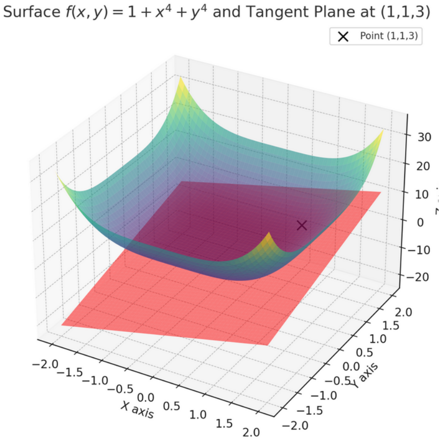
What is...the Jacobi criterion?

Or: Rank of a matrix

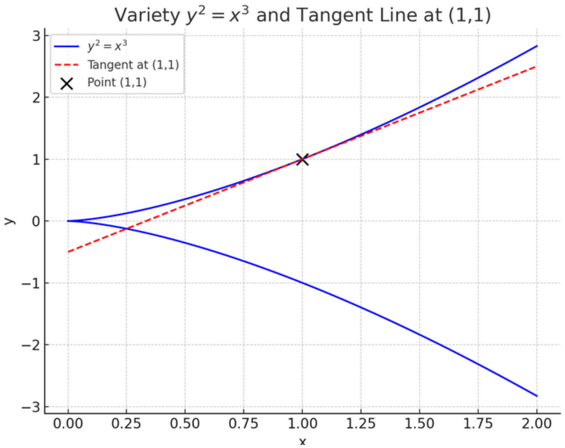
Smoothness in calculus

$f(x, y) = 1 + x^4 + y^4$:



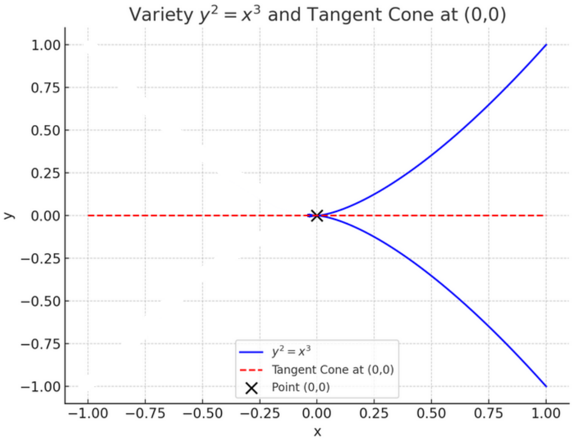
- Example For f compute $\mathbf{J} = (4x^3, 4y^3)$ which is of full rank at $(x, y) = (1, 1)$
- The tangent plane is then obtained using the Jacobi matrix $\mathbf{J}$
- Question Shouldn't we be able to do the same in AG?

Example



- Idea The Jacobi matrix of $V = V(y^2 - x^3 = 0)$ should be $\mathbf{J} = (-3x^2, 2y)$
- This is of full rank unless $x = y = 0$
- Full rank here means the dimension of the variety V itself

Example – continued



- Great The full rank criterion works for this example
- At $(x, y) \neq (0, 0)$ the matrix $\mathbf{J}$ is full rank and V is smooth
- At $(x, y) = (0, 0)$ the matrix $\mathbf{J}$ is not full rank and V is not smooth

For completeness: A formal statement

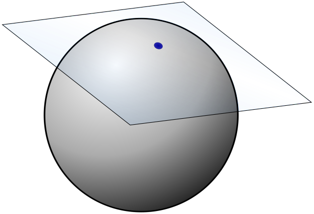
For some affine variety $V = (f_1, ..., f_m) \subset \mathbb{K}^n$:

(i) The Jacobi matrix is

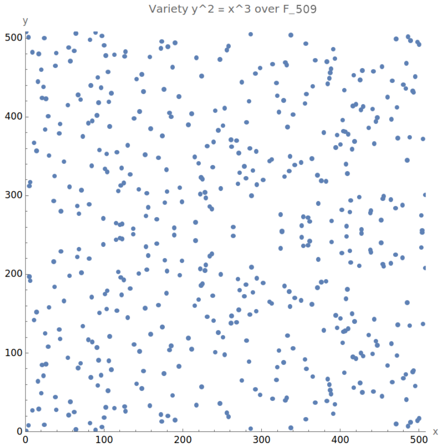
$$\mathbf{J_f} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

(ii) Jacobi criterion: V is smooth at $a \Leftrightarrow \mathrm{rk}\, \mathbf{J}(a)$ is maximal possible

- There is also a projective version of the Jacobi criterion
- Great This matches what happens in analysis



Nonsmooth smoothness



- This works over other fields not just $\mathbb{C}$
- Example Above is V over $\mathbb{F}_{509}$
- Crucial ‘smooth = no corners’ doesn’t work, but the algebraic version is fine

Thank you for your attention!

I hope that was of some help.