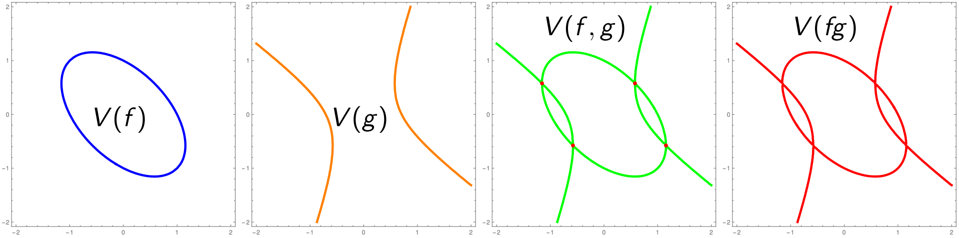
What are...ideals of sets?

Or: Enter, algebra!

Algebraic operations on V

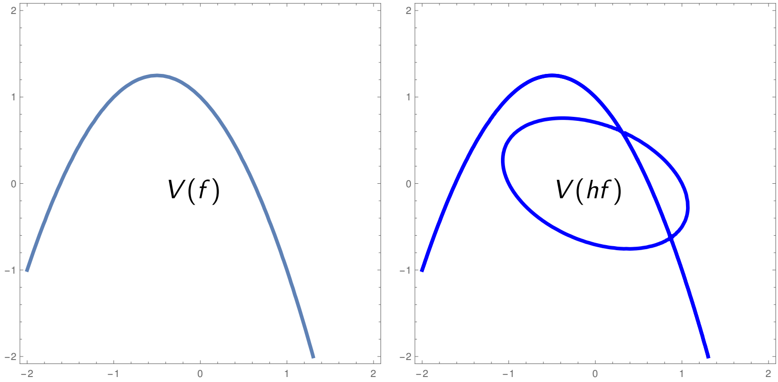


- Recall $V = V(P) =$ the points that are roots of all polynomials in P
- P gets smaller $\leftrightsquigarrow$ V gets bigger:

$$P \subset Q \Rightarrow V(P) \supset V(Q)$$
$$V(P \cup Q) = V(P) \cap V(Q)$$
$$V(PQ) = V(P) \cup V(Q)$$

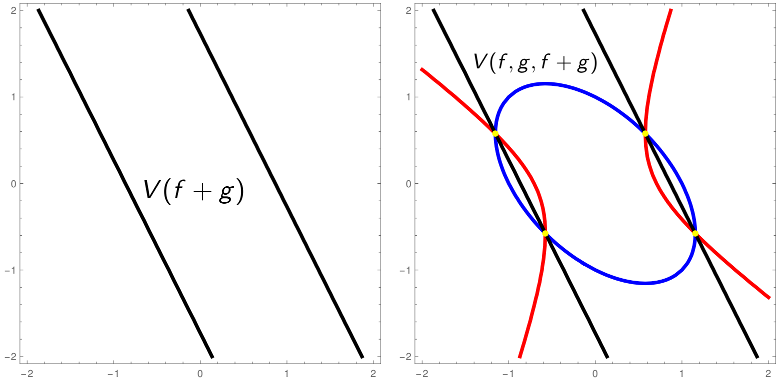
- This property is sometimes called contravariant

Ideal and varieties



- **Observation** If f vanishes on X, then so does hf for any h
- **Example** Above we have $V(f)$ and $V(hf)$ for certain f and h
  - Recall $V(P \cup Q) = V(P) \cap V(Q)$
  - So above $V(f) = V(f, hf)$
- **Idea** $V(P) = V(< P >) \Rightarrow$ study ideals associated to varieties and vice versa

Addition is a bit of a weird operation...



- Observation If f and g vanish on X, then so do $f+g$
- Example Above we have $V(f+g)$ and $V(f,g,f+g)$ for certain f and g
  - Recall $V(P \cup Q) = V(P) \cap V(Q)$
  - So above $V(f,g) = V(f,g,f+g)$
- Idea $V(P) = V(< P >) \Rightarrow$ study ideals associated to varieties and vice versa

For completeness: A formal statement

The ideal of X is

$$I = I(X) = \{f \in \mathbb{K}[x_1, ..., x_n] \mid f(v) = 0 \forall v \in X\}$$

where:

(i) $\mathbb{K}$ is some field

(ii) $X \subset \mathbb{K}^n$ is a collection of points

- I is an ideal in $\mathbb{K}[x_1, ..., x_n]$ and the "inverse" of V:

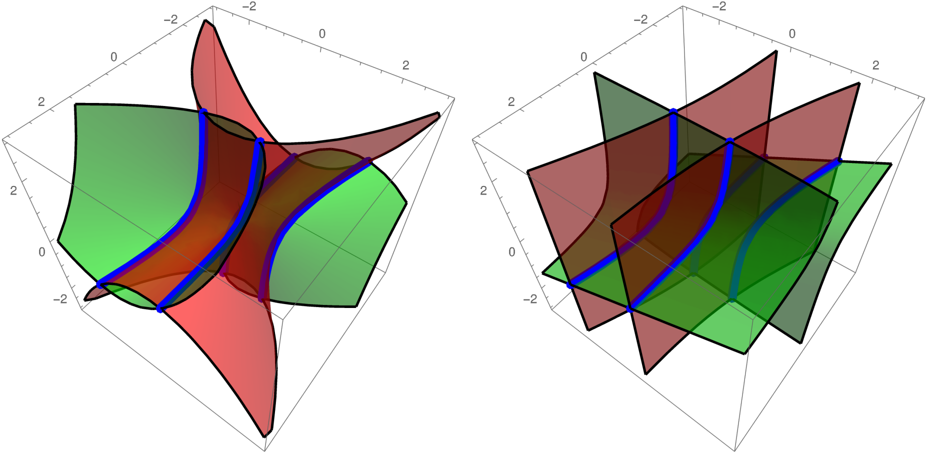
  $$\text{morally true: } V\big(I(X)\big) = X \text{ and } I\big(V(P)\big) = P$$

  We will see the correct statement in another video

- We finally see the first main flavor of algebraic geometry:

  $$\text{Varieties} \leftrightsquigarrow \text{ideals}$$

Another take on $V(f, g)$ (two different examples)



- $V(f)$ is green above; $V(g)$ is red above; $V(f, g)$ is blue above
- The two blue varieties are the same, the two green and red ones are different
- This corresponds to that ideals can have different generators

Thank you for your attention!

I hope that was of some help.