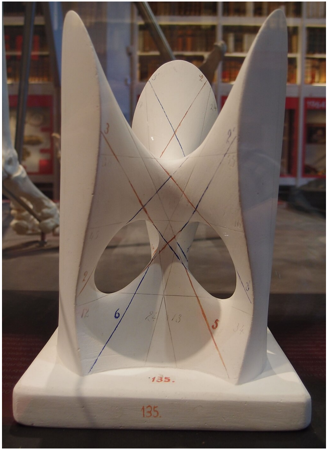
What are...27 lines?

Or: A cubic surprise!

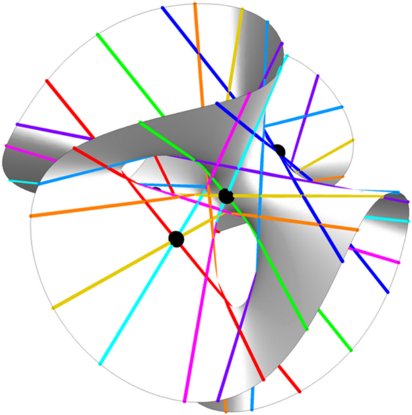
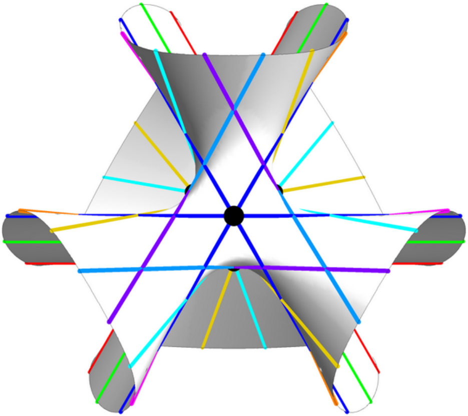
Clebsch, Klein and friends

$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$:



- **Above** Model of the Clebsch–Klein surface $V = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0) \subset \mathbb{P}^3$
- **Observe** the count of the lines on it
- **Today** In and around the origin of AG, based on this example

A surprising count

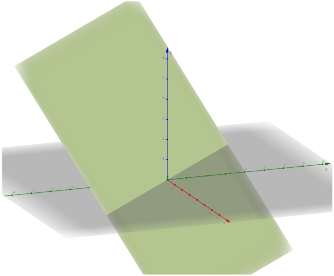
https://blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/

- Above Beautiful pictures from Egan (blog from Baez)
- Turns out that exactly 27 lines fit on V
- A lot of AG originates (∼**1850–1900**) trying to answer similar questions

Grassmann and friends

Grassmannian = higher lines

An element of $G(2,3)$:

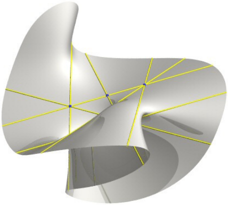


- Grassmannian $G(k,n)$ is the set of k-planes in $\mathbb{K}^n$ (here $k \in \{0, \dots, n\}$)
- Boring examples $G(0,n)$ and $G(n,n)$ are points
- Good example $G(1,n) = \mathbb{P}^{n-1}$, so we generalize projective space

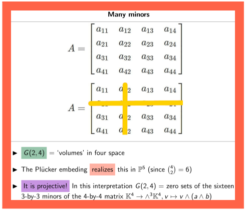
- Above The Grassmannian
- To find the lines one can use the Plücker coordinates of V in $G(2,4)$
- The ‘How many lines are on V?’ boils down to an equation count in $G(2,4)$

For completeness: A formal statement

Every smooth cubic contains exactly 27 lines



- Remarkable This result does not depend on the cubic!
- Proof? Uses the Grassmannian $G(2,4)$ and an equation count



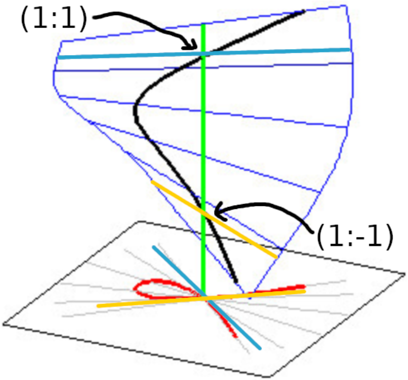
Many minors

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

- $G(2,4)$ = 'volumes' in four space
- The Plücker embeding realizes this in $\mathbb{P}^5$ (since $\binom{4}{2} = 6$)
- It is projective! In this interpretation $G(2,4)$ = zero sets of the sixteen 3-by-3 minors of the 4-by-4 matrix $\mathbb{K}^4 \to \wedge^3\mathbb{K}^4$, $v \mapsto v \wedge (a \wedge b)$

Alternative proof



- Fact Every smooth cubic is birationally equivalent to $\mathbb{P}^2$
- Using this we get: Every smooth cubic is a blow up of $\mathbb{P}^2$ at six points
- Under blow up, the lines correspond to 6 exceptional hypersurfaces, $15 = \binom{6}{2}$ lines through two of the points, $6 = \binom{6}{5}$ conics through five of the points

Thank you for your attention!

I hope that was of some help.