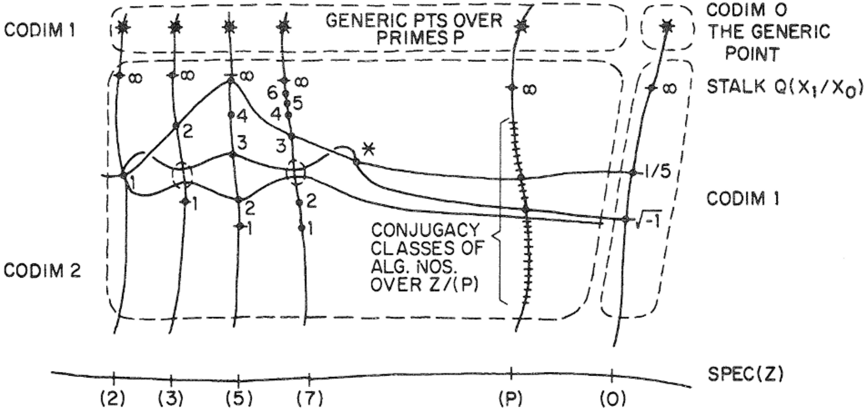
What are...schemes, take 1?

Or: Spec in action

Running example



- **Goal** Understand the picture above
- **Today** The bottom horizontal line
- The picture **is from** Mumford's "Lectures on Curves on an Algebraic Surface"

Reminder

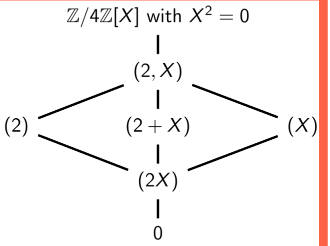
- We have **bijections**

$$\{\text{varieties}\} \overset{1:1}{\longleftrightarrow} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \mapsfrom P$$

- **Radical ideal** means $I = \sqrt{I}$
- One mild catch: the above are **order reversing**

- **Above** (A consequence of) Hilbert's Nullstellensatz
- **Idea** Define "varieties" on the right side
- **One upshot** is that we get rid of the underlying field

Reminder 2



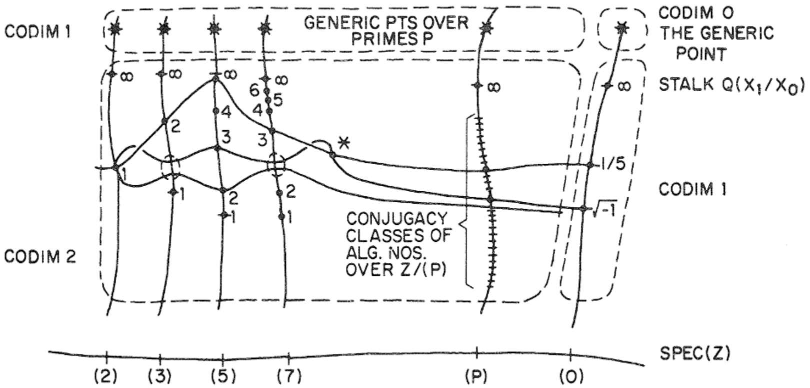
- $(2, X)$ is a maximal ideal
- $(2X)$ is a minimal ideal
- $(2, X)$ is a prime ideal
- (2), $(2 + X)$, (X) and $(2X)$ are principal ideals
- *etc.*

- Above Ideals in the ring $(\mathbb{Z}/4\mathbb{Z}[X])/(X^2)$
- Prime ideal $P \neq R$ and $(ab \in P \Leftrightarrow a \in P$ or $b \in P)$
- This mimics: $p \neq 1$ is prime $\Leftrightarrow$ ($p|ab$ implies $p|a$ or $p|b$)

For completeness: A formal statement

The spectrum $\mathrm{Spec}\, R$ of a commutative ring R is the set of all prime ideals

- Prime ideals in $\mathbb{Z}$: (0) or (p) for p prime
- $\mathrm{Spec}\, \mathbb{Z}$ is the bottom vertical line of



Zariski? Yes, we need it!

We have translations from geometry to algebra (for $\mathbb{K}$ algebraically closed)

(i) $V = W \cup U$ disconnected $\Rightarrow \mathbb{K}[V] \cong \mathbb{K}[W] \times \mathbb{K}[U]$

(ii) $V \neq \emptyset$ irreducible $\Leftrightarrow \mathbb{K}[V]$ is an integral domain

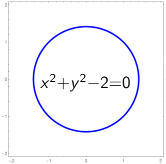
(iii) We have a 1:1 correspondence

$$\{\text{irreducible varieties} \neq \emptyset\} \xleftrightarrow{1:1} \{\text{prime ideals}\}$$

(iv) We have a 1:1 correspondence

$$\{\text{irreducible components of } V\} \xleftrightarrow{1:1} \{\text{minimal prime ideals in } \mathbb{K}[V]\}$$

(v) More... We will explore this!



$\mapsto (x^2 + y^2 - 2)$ prime ideal

- Right now $\operatorname{Spec} R$ is a set
- Scheme $= \operatorname{Spec} R$ plus a Zariski type topology
- Next time, motivated by the above, how does this work?

Thank you for your attention!

I hope that was of some help.