What are...schemes, take 2?

Or: Spec with a topology

Reminder

We have translations from geometry to algebra (for $\mathbb{K}$ algebraically closed)

(i) $V = W \cup U$ disconnected $\Rightarrow \mathbb{K}[V] \cong \mathbb{K}[W] \times \mathbb{K}[U]$

(ii) $V \neq \emptyset$ irreducible $\Leftrightarrow \mathbb{K}[V]$ is an integral domain

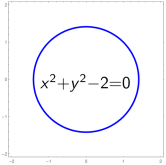
(iii) We have a 1:1 correspondence

$$\{\text{irreducible varieties} \neq \emptyset\} \xleftrightarrow{1:1} \{\text{prime ideals}\}$$

(iv) We have a 1:1 correspondence

$$\{\text{irreducible components of } V\} \xleftrightarrow{1:1} \{\text{minimal prime ideals in } \mathbb{K}[V]\}$$

(v) More... We will explore this!



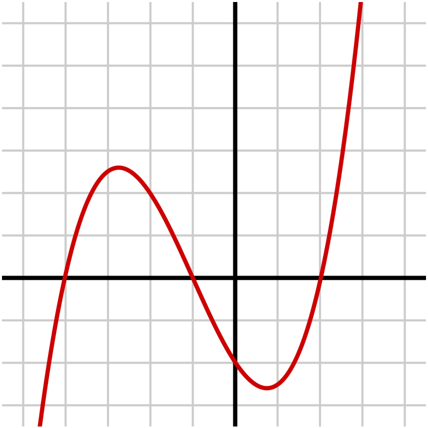
$\mapsto (x^2 + y^2 - 2)$ prime ideal

- Right now $\operatorname{Spec} R$ is a set
- Scheme $= \operatorname{Spec} R$ plus a Zariski type topology
- Next time, motivated by the above, how does this work?

$\mathbb{C}[x]$ and $\mathbb{R}[x]$

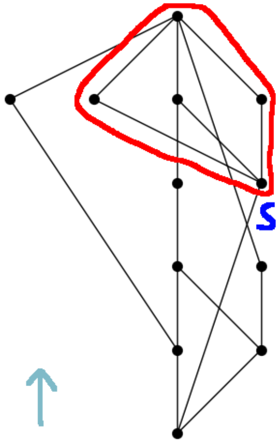
degree 3

factors :



- Often prime = can decompose further, and prime ideal = spanned by primes
- Example In $\mathbb{C}[x]$ the prime ideals are (0) and $(x+a)$, $a \in \mathbb{C}$
- Example In $\mathbb{R}[x]$ the prime ideals are (0) and $(x+a)$ and (x^2+ax+b) for $a^2-4b<0$, $a,b \in \mathbb{R}$

The ones that are bigger



- Above Prime ideals of a ring form a lattice using inclusion
- The ring is left implicit (does not matter right now)
- The prime ideals above (bigger than) S are circled

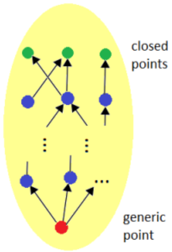
For completeness: A formal statement

Define a topology on $\mathrm{Spec}\, R$ whose closed sets are

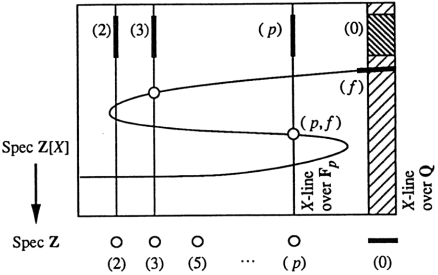
$$V(S) = \{T \in \mathrm{Spec}\, R | T \supset S\}$$

Careful: this is again defined via the closed sets

- This is the Zariski topology on $\mathrm{Spec}\, R$; and indeed one can check that this defines a topology
- Two extremes A point $\{S\}$ is closed $\Leftrightarrow$ S is a maximal ideal; a point $\{S\}$ is generic, by definition, if the closure is all of R



Running example 2



1. (0).
2. $(f(X))$, where $f(X)$ is an irreducible polynomial.
3. (p), where p is a prime number.
4. $(p, f(X))$, where p is a prime number and $f(X)$ is an irreducible polynomial modulo p.

- Bottom $\operatorname{Spec}\mathbb{Z}$ – we understand this one
- Whole picture $\operatorname{Spec}\mathbb{Z}[x]$; prime ideals are above
- The picture is from Reid's "Undergraduate Commutative Algebra"

Thank you for your attention!

I hope that was of some help.