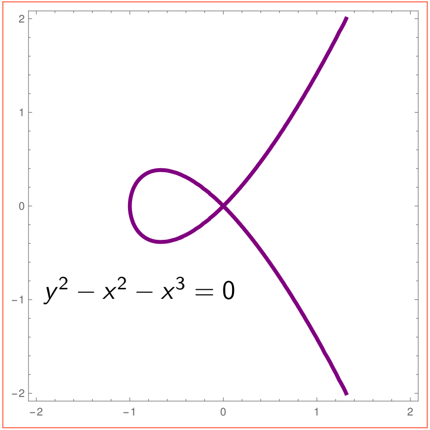
What are...schemes, take 3?

Or: How to evaluate at ideals?

Affine schemes



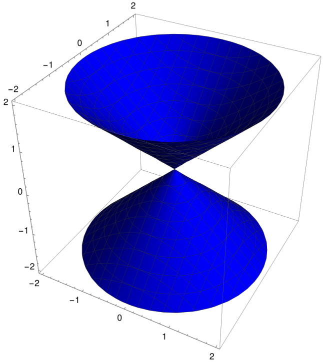
$\leftrightsquigarrow \mathbb{K}[x,y]/(y^2 - x^2 - x^3)$

- (Affine) Scheme $= \operatorname{Spec} R$ plus a Zariski type topology
- Idea Instead of a variety V we look at its coordinate ring $\mathbb{K}[V]$: take $R = \mathbb{K}[V]$ and $\operatorname{Spec} R$ is an affine scheme
- The scheme setting is more general (*e.g.* not just quotients of polynomials)

Projective schemes

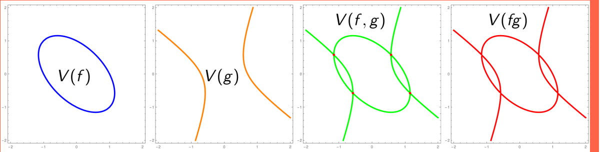
Homogenization

$x^2 + y^2 = z^2 \leftrightsquigarrow$



- Projective scheme $= \mathrm{Proj}\, R$ (homogeneous prime ideals for a graded ring R) plus a Zariski type topology
- Why? Recall that projective varieties come from homogeneous polynomials
- The idea is then the same as for affine schemes, just graded

Almost everything generalizes



- We have bijections

$$\{\text{varieties}\} \stackrel{1:1}{\longleftrightarrow} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \leftarrow P$$

- Radical ideal means $I = \sqrt{I}$
- One mild catch: the above are order reversing

- We have scheme varieties and ideals:

$$V = V(Y) = \{P \in \mathrm{Spec}\, R \mid f(P) = 0 \forall f \in Y\} \qquad I = I(X) = \{f \in R \mid f(P) = 0 \forall P \in X\}$$

where $f(P) = 0$ is explained on the next page

- Scheme Nullstellensatz Essentially the same with the above constructions

For completeness: A formal statement

Evaluation in a ring R and prime ideal P

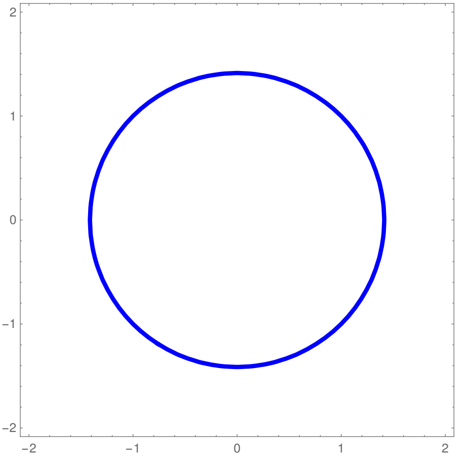
- R/P is an integral domain
- Residue field $K = K(P)$ of $\mathrm{Spec}\, R$ at P is quotient field of R/P
- For $f \in R$ define $f(P)$ to be the image of f under $R \to R/P \to K$
  Ring-theoretic evaluation

- **Theorem** With the above, everything that we have seen so far has a scheme theoretic incarnation (not spelled out anymore)
- **To keep in mind** ($\mathbb{K}$ is alg. closed)

$$\begin{array}{ccc}
\{\text{affine varieties}\} & \longleftrightarrow & \{\text{fin. gen., nilpotent-free } \mathbb{K}\text{-algebras}\} \\
\downarrow & & \downarrow \\
\{\text{affine schemes}\} & \longleftrightarrow & \{\text{commutative rings with identity}\}
\end{array}$$

But why would a geometer care?

$V(x^2+y^2+1) = V\left((x^2+y^2+1)^2\right) \leftrightsquigarrow$



- Problem Varieties cannot see multiplicities
- Baby example $V(x) = V(x^2)$ is just a point; the multiplicity is lost
- But the schemes are different: one is $\mathbb{C}[x]/(x)$ the other is $\mathbb{C}[x]/(x^2)$

Thank you for your attention!

I hope that was of some help.