What are...regular function for schemes?

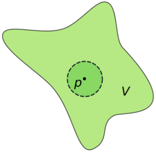
Or: Division without a field

What to do without a field?

Regular functions:

- V affine variety, $U \subset V$ open
- $\phi\colon U \to \mathbb{K}$ is regular if $\phi = f_p/g_p$ on U_p for all $p \in U$ for some $f_p, g_p \in \mathbb{K}[V]$

- The "for all $a \in U$" makes the condition local



- Above Regular functions on an affine variety defined over a ground field $\mathbb{K}$
- Scheme $= \operatorname{Spec} R$ plus a Zariski type topology – no ground field!
- Question What does that actually mean without a field?

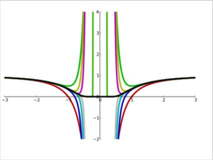
Regular functions ring theoretical

For V affine variety, $f \in \mathbb{K}[V]$ we have:

$$\mathcal{O}_V(D(f)) \cong \mathbb{K}[V]_f$$

where $\mathbb{K}[V]_f$ = localization of $\mathbb{K}[V]$ along $S = \{f, f^2, f^3, \ldots\}$

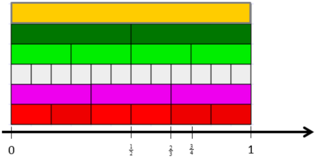
- Here $D(f) = V \setminus V(f) = \{v \in V | f(v) \neq 0\}$ are distinguished open sets
- Regular to coordinate functions $\rightsquigarrow$ Laurent to usual polynomials



e^{-1/x^2} and Laurent approximations

- Important object Ring of regular functions $\mathcal{O}$
- Observation This ring is the same as localization of the coordinate ring
- Idea Take this as the schemes theoretic definition

Localizations and regular functions

<table>
<tr><th></th><th>Abstract</th><th>Incarnation</th></tr>
<tr><td>Numbers</td><td>3</td><td> or…</td></tr>
<tr><td>Fractions</td><td>$\frac{r}{s}$</td><td>

or $\frac{X^2+1}{X^3+1}$ or…</td></tr>
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Fractions, a.k.a. "a part of a whole", work in great generality

- Take an open subset U of $\operatorname{Spec} R$
- A regular function should be a collection $(\phi_P \in R_P)_{P \in U}$
- Here P are prime ideals so we can localize R at P and ge the localization R_P

For completeness: A formal statement

Regular functions:

- $\operatorname{Spec} R$ affine scheme, $U \subset \operatorname{Spec} R$ open
- $(\phi_P \in R_P)_{P \in U}$ is regular if, for all Q in an open subset $U_P \subset U$ containing P, $\phi_P = f/g \in R_Q$ where $f, g \in R$, $g \notin Q$

- The story than goes through as for "usual" regular functions

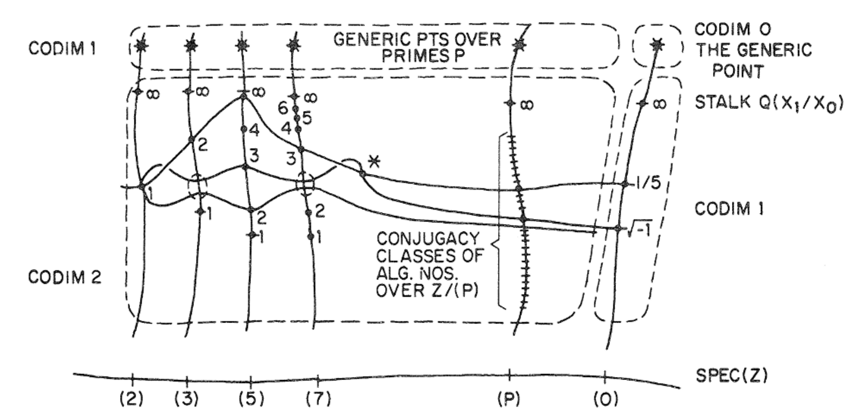
For V affine variety, $f \in \mathbb{K}[V]$ we have:

$$\mathcal{O}_V(D(f)) \cong \mathbb{K}[V]_f$$

where $\mathbb{K}[V]_f$ = localization of $\mathbb{K}[V]$ along $S = \{f, f^2, f^3, \dots\}$

- The "for all $Q \in U_P$" makes the condition local
- We get an important object in AG: the ring of regular functions is the sheaf $\mathcal{O}_{\operatorname{Spec} R}(U)$ of regular functions on U

Running example – back to it



- $\operatorname{Spec}\mathbb{Z}$ is an affine scheme, consider $D(6) = \{(0)\} \cup \{(p) | p \neq 2, 3\}$
- $\phi = 5/6$ is a regular function on $D(6)$ seen in $\mathbb{Z}_{\{6^i | i=1,2,\dots\}}$
- How? $\phi((0)) = 5/6 \in \mathbb{Q}$, $\phi((p)) = 5/6 \in \mathbb{Z}/p\mathbb{Z}$ (e.g. $= 0$ for $p = 5$)

Thank you for your attention!

I hope that was of some help.