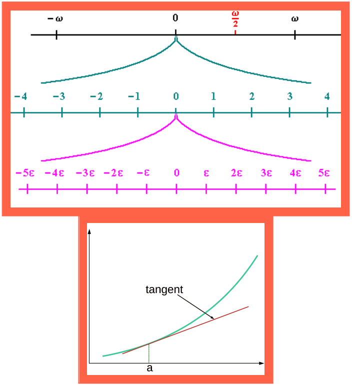
What are...morphisms of schemes?

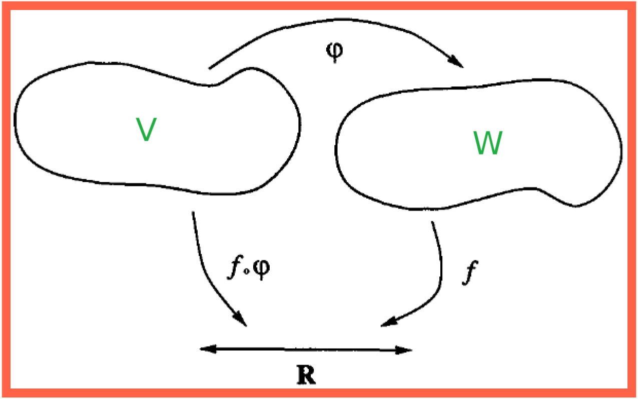
Or: Functions that are not determined by their input

Dual numbers again



- **Dual numbers** $R = \mathbb{K}[X]/(X^2)$ (think of X as being a linear infinitesimal) with one prime ideal $P = (X)$
- **Regular functions** on $\operatorname{Spec} R$ are of the form $\phi = a + b \cdot X$ for $a, b \in \mathbb{K}$
- For schemes, regular functions are **not determined** by their inputs since $\phi(P) = a$ for all ϕ

This doesn't generalize immediately



- Recall that we think of V as the pair $(V, \mathcal{O}_V)$ (V and its regular functions)
- It then made sense to demand that the pullback $\varphi^* f = f \circ \varphi$ is a regular function for all regular functions $f\colon U \subset W \to \mathbb{K}$
- Problem For schemes, regular functions are not determined by their evaluation, so this is probably the wrong thing to do

Assume existence

- Recall $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces if:
  (i) It is continuous
  (ii) For $f \in \mathcal{O}_Y(U)$ we have $\varphi^* f = f \circ \phi \in \mathcal{O}_X(f^{-1}(U))$
- For schemes we assume that φ^* exist (whatever it is)

For completeness: A formal statement

$\varphi\colon (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \to (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ is a morphism of schemes if:

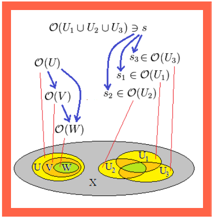
(i) It is continuous

(ii) For every open subset $U \subset \operatorname{Spec} S$ a ring homomorphism
$\phi_U^*\colon \mathcal{O}_{\operatorname{Spec} S}(U) \to \mathcal{O}_{\operatorname{Spec} R}(\phi^{-1}(U))$ Pullback
such that:

(I) The pullback maps are compatible with restrictions

(II) A condition on maximal ideals (omitted)

- Note that we assume existence of the pullback
- The story than goes through as for "usual" morphisms of varieties



Great, the expected generalization

For affine varieties V, W (over alg. closed $\mathbb{K}$) there is a bijection

$$\{\text{maps } V \to W\} \xleftrightarrow{1:1} \{\text{homomorphisms } \mathbb{K}[W] \to \mathbb{K}[V]\}, f \mapsto f^*$$

Moreover, we have

$$\{\text{affine varieties}\}/\text{iso.} \xleftrightarrow{1:1} \{\text{fin. gen. reduced } \mathbb{K}\text{-algebras}\}/\text{iso.}$$

- Fin. gen reduced $\mathbb{K}$-algebras $\rightsquigarrow$ quotient of a polynomial ring

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- Above For varieties
- For schemes $\mathrm{Spec}\, R, \mathrm{Spec}\, S$ there is a bijection

$$\{\text{maps } \mathrm{Spec}\, R \to \mathrm{Spec}\, S\} \xleftrightarrow{1:1} \{\text{homomorphisms } S \to R\}, f \mapsto f^*$$

Moreover, we have

$$\{\text{affine schemes}\}/\text{iso.} \xleftrightarrow{1:1} \{\text{rings}\}/\text{iso.}$$

Thank you for your attention!

I hope that was of some help.