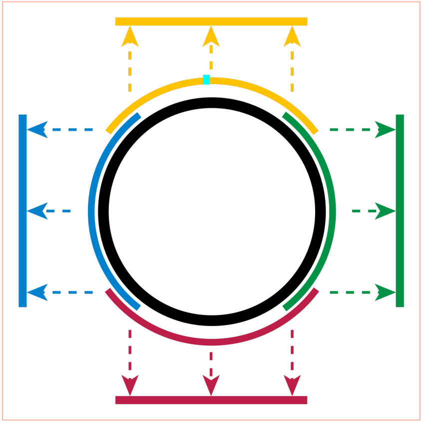
What are...schemes, take 4?

Or: Schemy summary

Patchworks



- Manifold = something that is patched together from ‘discs’
- Variety = something that is patched together from affine varieties
- Scheme = something that is patched together from affine schemes

Some conditions

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent, with

$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \; B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \; B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Actually, scheme = patchwork plus a few conditions
- Reduced (condition 1) No nilpotent elements in $\mathcal{O}_{\operatorname{Spec} R}(U)$ for open U
- Finite type + separated (conditions 2 + 3) A finiteness condition and a Hausdorff condition

Varieties are schemes

For affine varieties V, W (over alg. closed $\mathbb{K}$) there is a bijection

$$\{\text{maps } V \to W\} \overset{1:1}{\longleftrightarrow} \{\text{homomorphisms } \mathbb{K}[W] \to \mathbb{K}[V]\}, f \mapsto f^*$$

Moreover, we have

$$\{\text{affine varieties}\}/\text{iso.} \overset{1:1}{\longleftrightarrow} \{\text{fin. gen. reduced } \mathbb{K}\text{-algebras}\}/\text{iso.}$$

- Fin. gen reduced $\mathbb{K}$-algebras $\rightsquigarrow$ quotient of a polynomial ring

---

- Affine varieties are certain affine schemes
- Varieties are certain schemes:

$$\{\text{varieties over } \mathbb{K}\}/\text{iso.} \overset{1:1}{\longleftrightarrow} \{\text{separated reduced schemes of finite type over } \mathbb{K}\}/\text{iso.}$$

For completeness: A formal statement

Let's summarize:

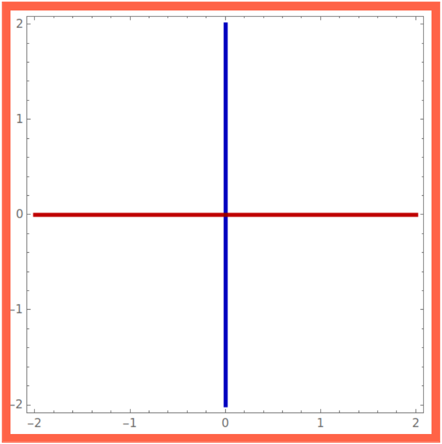
(i) $\operatorname{Spec} R$ affine scheme = prime ideals in R with Zariski topology

(ii) $\operatorname{Proj} R$ projective scheme = homo. prime ideals in R with Zariski topology

(iii) General scheme = has a covering with affine schemes

- The story then goes through as for varieties, but beefed-up:
  - We have regular functions $\mathcal{O}$ (localizations) and ringed spaces
  - Schemes correspond to general rings
  - We can distinguish $x = 0$ and $x^2 = 0$

The geometry of fat points



- Classical varieties do not give any higher order points
- Double point arise from $\mathrm{Spec}\,\mathbb{K}[X]/(X^2)$
- Triple point arise from $\mathrm{Spec}\,\mathbb{K}[X]/(X^3)$ or $\mathrm{Spec}\,\mathbb{K}[X,Y]/(X^2,Y^2,XY)$

Thank you for your attention!

I hope that was of some help.