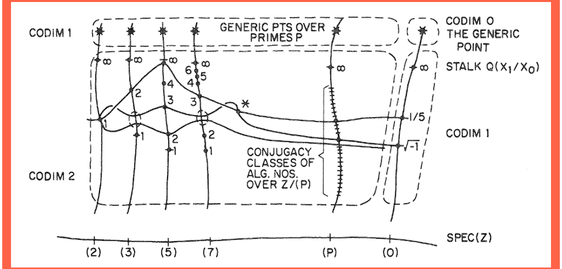
What are...schemes, take 5?

Or: Back to the main example

Mumford's $\operatorname{Spec} \mathbb{Z}[x]$

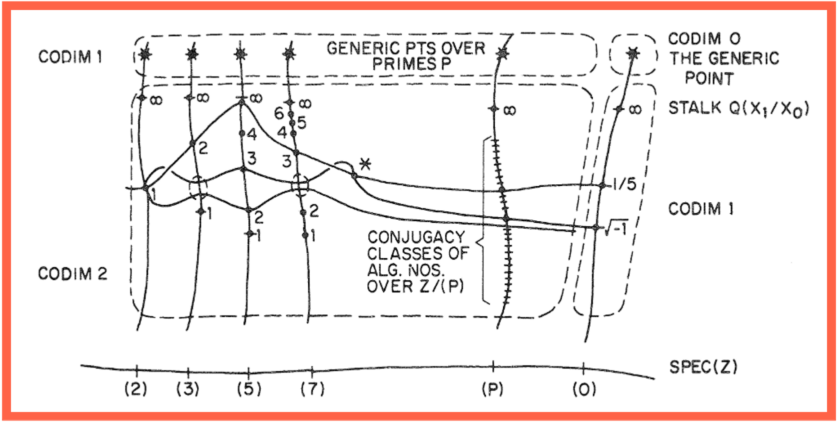


prime ideals:

1. (0).
2. $(f(X))$, where $f(X)$ is an irreducible polynomial.
3. (p), where p is a prime number.
4. $(p, f(X))$, where p is a prime number and $f(X)$ is an irreducible polynomial modulo p.

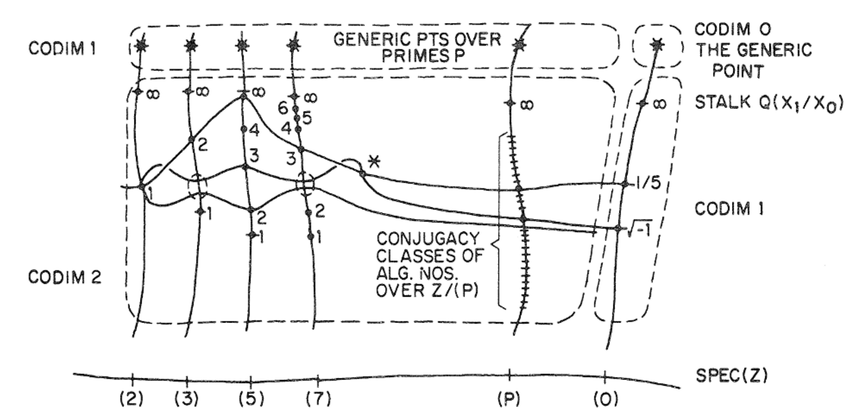
- Above Mumford's drawing of $\operatorname{Spec} \mathbb{Z}[x]$ over $\operatorname{Spec} \mathbb{Z}$
- The diagram to emphasize the interplay between the arithmetic (primes) and geometry (behavior of polynomials)
- Today Let us understand the picture!

What is up with $1/5$?



- 1/5 corresponds to the irreducible polynomial $5x - 1$
- Observation This polynomial has a solution unless we are in $\mathbb{Z}/5\mathbb{Z}$ (where we use ∞ as a solution), e.g. $1/5 = 2$ in $\mathbb{Z}/3\mathbb{Z}$
- Think Generic points are $\mathbb{Q}_p$ or $\mathbb{Q}$

What is up with $\sqrt{-1}$?



- $\sqrt{-1}$ corresponds to the irreducible (over $\mathbb{Z}$) polynomial $x^2 + 1$
- Observation Whether this polynomial is irreducible modulo p depends on p
- Example $x^2 + 1 = (x - 2)(x - 3)$ over $\mathbb{Z}/5\mathbb{Z}$ but this is irreducible over $\mathbb{Z}/3\mathbb{Z}$

For completeness: A formal statement

Not really formal, but anyway:

- $\operatorname{Spec}\mathbb{Z}[x]$ combines the structure of $\operatorname{Spec}\mathbb{Z}$ with the algebraic properties of polynomials
- The picture helps to track how global information (like irreducibility over $\mathbb{Q}[x]$) localizes to specific primes

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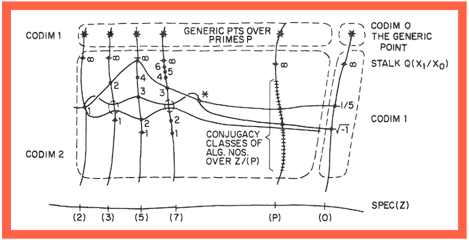
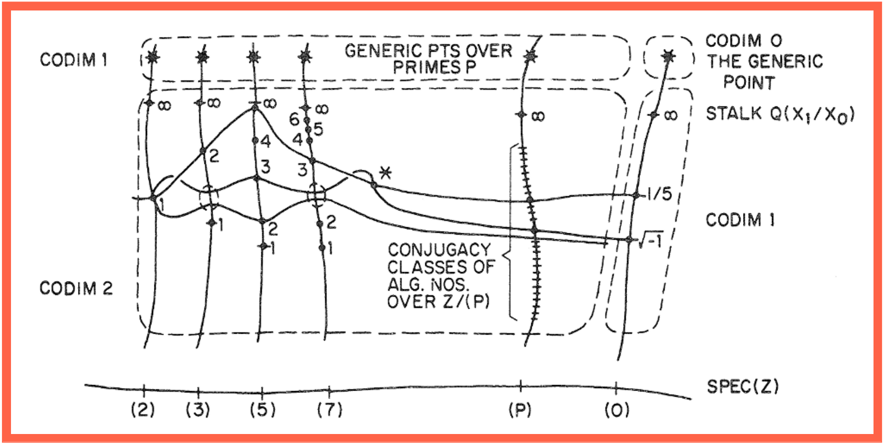
- There is also a famous flipped version by Manin http://www.neverendingbooks.org/manins-geometric-axis/



- Codimension $=$ maximal number of irreducible closed subsets that fit between $Y \subset X$

What are the dashed boxes now?



- Codimension 0 The generic point (0)
- Codimension 1 Ideals $(f(x))$ or (p)
- Codimension 2 Ideals $(p, f(x))$ (think: irreducible polynomials modulo p)

Thank you for your attention!

I hope that was of some help.