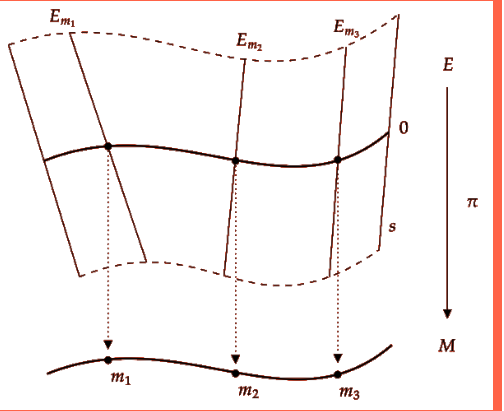
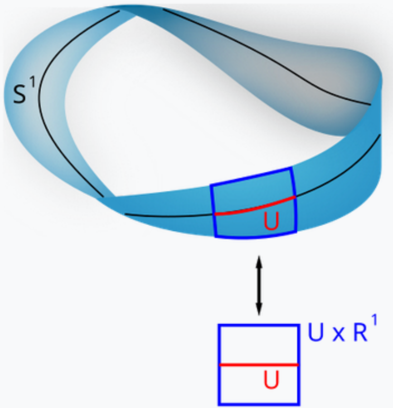
What is...a tautological sheaf?

Or: Tautological sheaves are tautological sheaves

Sheaves of modules "=" vector bundles



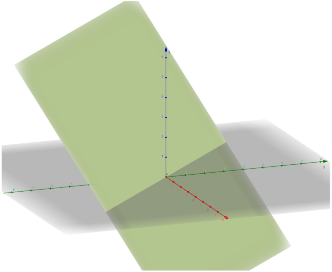


- Vector bundle = a linear algebra generalization of a cylinder
- Idea To every point of the space we attach a vector space so that these fit together to form another space of a desired type (manifold, or variety, or...)
- Sheaves of modules and vector bundles are very similar

Grassmannians again

Grassmannian = higher lines

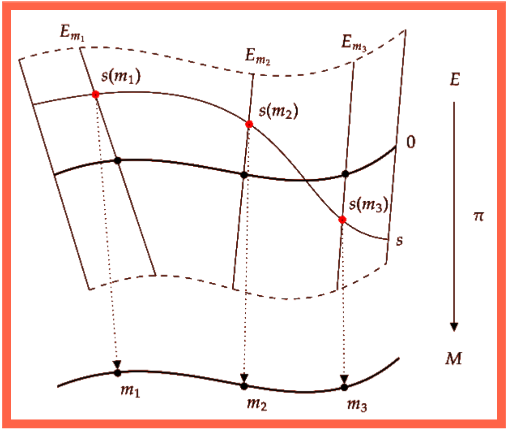
An element of $G(2,3)$:



- Grassmannian $G(k,n)$ is the set of k-planes in $\mathbb{K}^n$ (here $k \in \{0,\dots,n\}$)
- Boring examples $G(0,n)$ and $G(n,n)$ are points
- Good example $G(1,n) = \mathbb{P}^{n-1}$, so we generalize projective space

- Points of the Grassmannian are vector spaces
- They also vary continuously
- Think There should be a tautological way to get a vector bundle

Sheaves = vector bundles?



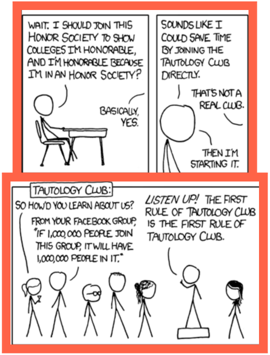
- Section = a way to put the original space into its vector bundle
- Sheaf of sections is a sheaf associated to a vector bundle; every locally free (lf) sheaf arises in this way
- Lf sheaf "=" lf $\mathcal{O}_V$-module = locally isomorphic to a free $\mathcal{O}_V$-module

For completeness: A formal statement

Tautological bundle/sheaf (for $k = 1$ via AG on the next slide):

- The tautological bundle on $G(k, n)$
- $k = 1$: tautological line bundle on $\mathbb{P}^1$

- Tautology xkcd can explain much better than me what a tautology is:



- There are some subtleties but nothing serious, so we ignore them

The same target, but colored

- Recall For a projective variety, $\mathcal{O}_V$ is a graded ring
- As a sheaf of $\mathcal{O}_V$-modules we take $\mathcal{O}_V$ itself with a shifted grading
- $\mathcal{O}(d)$ (shift grading by d) and $\mathcal{O}(-1)$ = tautological bundle over $\mathbb{P}^1$

Thank you for your attention!

I hope that was of some help.