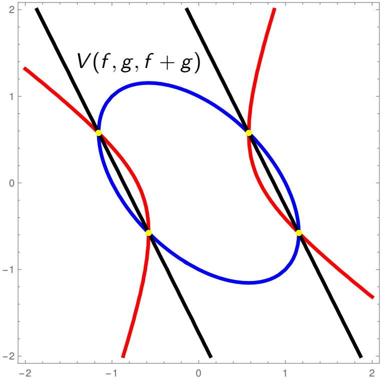
What is...Hilbert's Nullstellensatz?

Or: The zero-locus-theorem

V and I



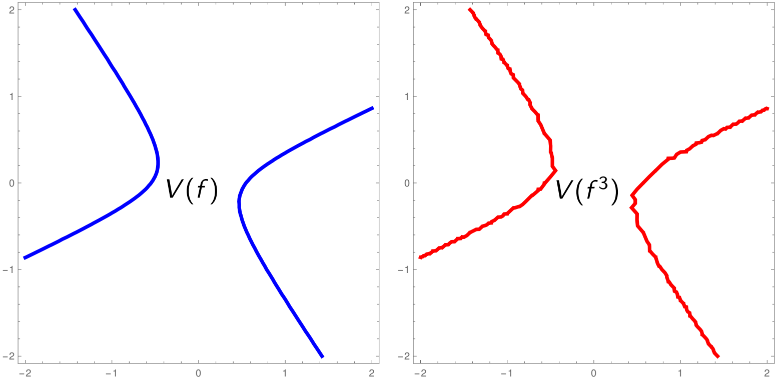
- Recall, we had varieties and ideals:

$V = V(P) = \{v \in \mathbb{K}^n \mid f(v) = 0 \forall f \in P\}$ $\quad I = I(X) = \{f \in \mathbb{K}[x_1, ..., x_n] \mid f(v) = 0 \forall v \in X\}$

- $V\big(I(X)\big) = X$ is fairly easy to see

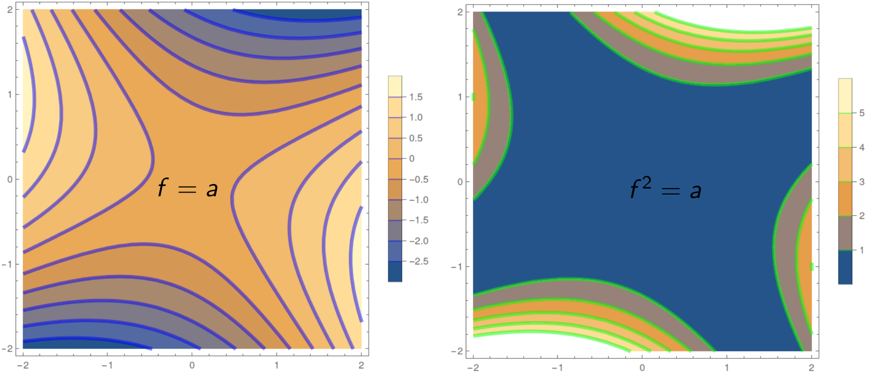
- Question What about the converse?

Varieties and powers



- Observation The vanishing sets of f and f^n are the same
- Example Above we have a (very low resolution :-)) hyperbola given by $(1/3x^2 - 1/2xy - 1/2y^2 - 0.1)^{1 \text{ or } 3} = 0$
- Mild catch This really needs the complex numbers

Real plots get now a bit tricky



- Above The level sets of f and f^2
- Observation Up scaling, $f = a$ and $f^2 = a$ agree for $a > 0$ but not for $a \leq 0$
- Idea We probably want to work with complex numbers instead

For completeness: A formal statement

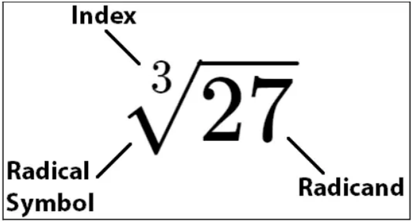
We have Hilbert's Nullstellensatz:

(i) $V\big(I(X)\big) = X$ An inverse

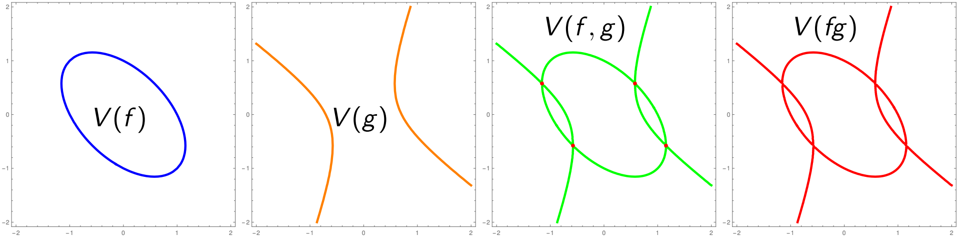
(ii) $I\big(V(P)\big) = \sqrt{P}$ Almost an inverse

Here our ground field is algebraically closed (e.g. $\mathbb{K} = \mathbb{C}$)

- $\sqrt{P} = \{f \in \mathbb{K}[x_1, ..., x_n] \mid f^k \in P \text{ for some } k \in \mathbb{N}\}$ is the so-called radical
- The name comes from the old word for root:



Identifying varieties and ideals



- We have bijections

$$\{\text{varieties}\} \overset{1:1}{\longleftrightarrow} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \leftarrow\!\shortmid P$$

- Radical ideal means $I = \sqrt{I}$

- One mild catch: the above are order reversing

Thank you for your attention!

I hope that was of some help.