What are...quasi-coherent sheaves?

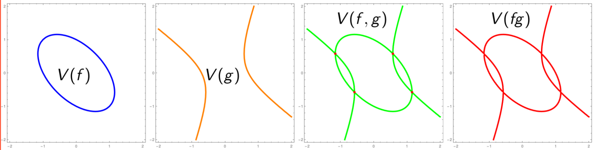
Or: Translate to algebra, please

All I know about AG

- Recall Algebraic geometry (AG) loves rings
- More precisely, AG wants to translate geometry question to algebra questions
- Problem Sheaves of modules are too general and do not resonate well with algebra

R modules = linear algebra

Identifying varieties and ideals



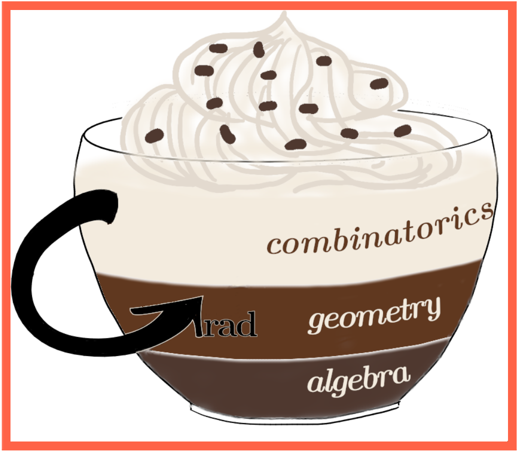
- We have bijections

$$\{\text{varieties}\} \xleftrightarrow{1:1} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \leftarrow P$$

- Radical ideal means $I = \sqrt{I}$
- One mild catch: the above are order reversing

- Above Hilbert's Nullstellensatz as a prototypical example how to go between geometry and algebra
- Idea Every R-module should give us a sheaf of modules that behaves "very algebraically" and can be studied using module theory

Sheaves associated to R-modules



- Recall $\mathcal{O}_{\operatorname{Spec} R}$ = sheaf of regular functions
- In this definition replace R by M for some R-module M
- Result $\tilde{M}$ = sheaf of regular functions on M (sheaf of R-modules)

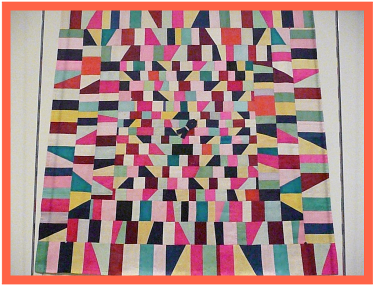
For completeness: A formal statement

A sheaf of modules $\mathcal{F}$ is quasi-coherent if:

(i) There is an affine open cover $\{U_i = \mathrm{Spec}\, R_i\}$ of V

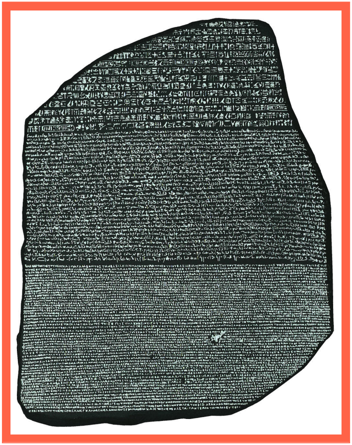
(ii) The restrictions $\mathcal{F}_{U_i}$ are isomorphic to some $\tilde{M}_i$

- This is like a patchwork of $\tilde{M}_i$



- Fact For any open subset U the restriction $\mathcal{F}_U$ is isomorphic to some $\tilde{M}_U$
- Coherent $=$ same but for finitely generated M_i

Translate to module theory



- Most statements about quasi-coherent sheaves can be translated into algebra
- Example For M and N, both R-modules, there is a bijection

$$\{\text{morphisms of sheaves } \tilde{M} \to \tilde{N}\} \xleftrightarrow{1:1} \{R\text{-module morphisms } M \to N\}$$

Thank you for your attention!

I hope that was of some help.