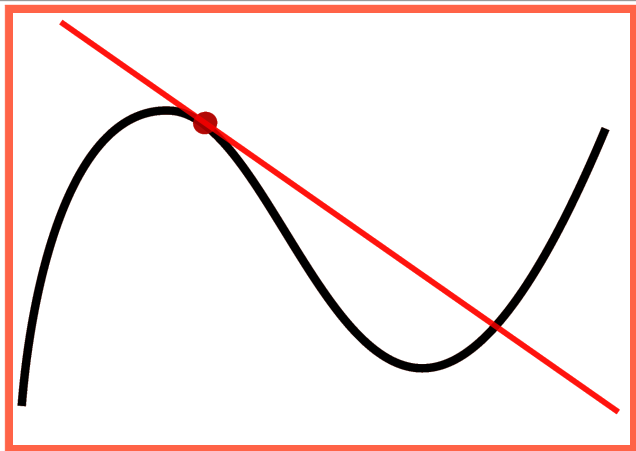


What are...Kähler differentials, part 1?

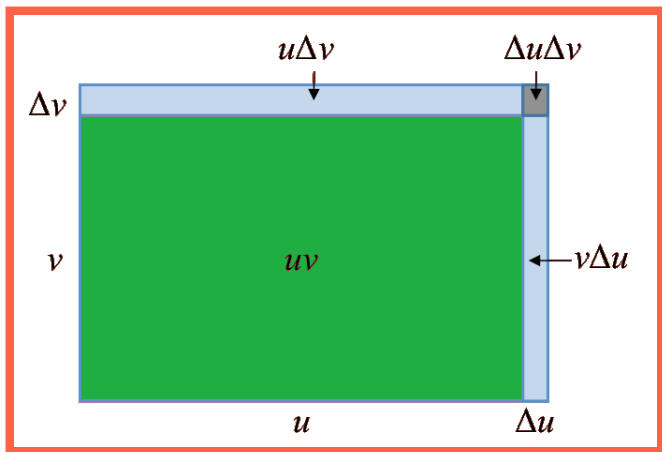
Or: Algebraic calculus

Tangents and friends



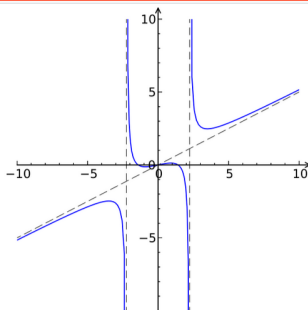
-
- ▶ **Differentials** are one of the most important ideas in mathematics
 - ▶ **Semiclassical** they are defined using limits – bad for algebra
 - ▶ **Idea** Let's get rid of limits and make this algebraic

Infinitesimals



- ▶ Above The classical proof of the product rule $(fg)' = fg' + f'g$
- ▶ Idea Instead of limits, use differential as formal symbols satisfying formal rules

Rules for free



Rational function of degree 3, with a graph

of degree 3: $y = \frac{x^3 - 2x}{2(x^2 - 5)}$

- ▶ **Recall** For polynomials the product rule and $x' = 1$ imply the power rule
- ▶ **Recall** For polynomials the power rule, product rule and $(f + g)' = f' + g'$ imply the chain rule
- ▶ **Recall** For rational functions even the quotient rule follows from the others

For completeness: A formal statement

For a \mathbb{K} -algebra R let Ω_R be the free R -module generated by df for $f \in R$ modulo:

- (i) $df = 0$ for $f \in \mathbb{K}$ (Scalars do not grow)
- (ii) $d(f + g) = df + dg$ (Linear)
- (iii) $d(fg) = f(dg) + (df)g$ (Product rule)


Kähler differentials

- ▶ In the localization the **quotient rule is free**, so this extends to regular functions

Regular functions:

- ▶ V affine variety, $U \subset V$ open
- ▶ $\phi: U \rightarrow \mathbb{K}$ is regular if $\phi = f_p/g_p$ on U_p for all $p \in U$ for some $f_p, g_p \in \mathbb{K}[V]$

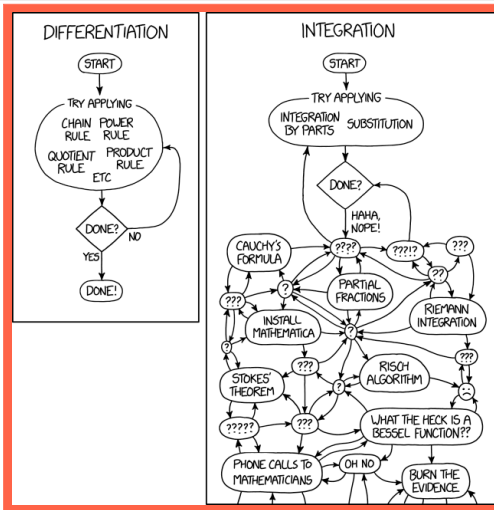
- ▶ The "for all $a \in U$ " makes the condition **local**



- ▶ We get an important object in AG: the **ring of regular functions** is the \mathbb{K} -algebra $\mathcal{O}_V(U)$ of regular functions $\phi: U \rightarrow \mathbb{K}$ with pointwise operations
- ▶ What should be written (but isn't because its a mouthful) is "For every $a \in U$ there exist $f_p, g_p \in \mathbb{K}[V]$ with $g_p(a) \neq 0$ and $\phi(x) = f_p(x)/g_p(x)$ for all x in an open subset $U_p \subset U$ with $a \in U_p$ "

- ▶ **As we will see**, this is the key for using differentials in algebraic geometry

We have everything



- ▶ Differentiation “=” Product, chain, power and quotient rule
- ▶ XKCD (“=” experience) confirms that ☺

Thank you for your attention!

I hope that was of some help.