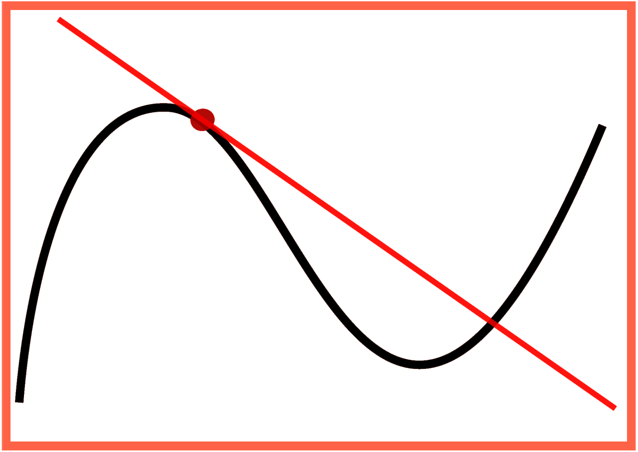
What are...Kähler differentials, part 1?

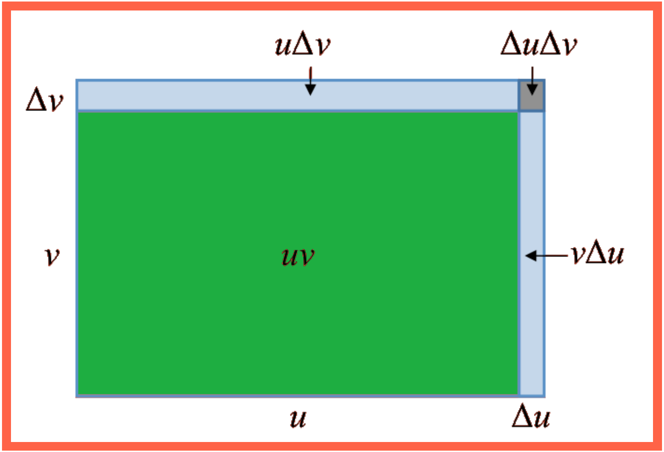
Or: Algebraic calculus

Tangents and friends



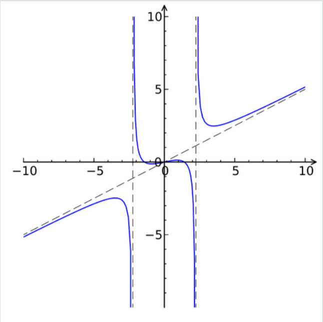
- Differentials are one of the most important ideas in mathematics
- Semiclassical they are defined using limits – bad for algebra
- Idea Let's get rid of limits and make this algebraic

Infinitesimals



- Above The classical proof of the product rule $(fg)' = fg' + f'g$
- Idea Instead of limits, use differential as formal symbols satisfying formal rules

Rules for free



Rational function of degree 3, with a graph of degree 3: $y = \frac{x^3 - 2x}{2(x^2 - 5)}$

- Recall For polynomials the product rule and $x' = 1$ imply the power rule
- Recall For polynomials the power rule, product rule and $(f + g)' = f' + g'$ imply the chain rule
- Recall For rational functions even the quotient rule follows from the others

For completeness: A formal statement

For a $\mathbb{K}$-algebra R let Ω_R be the free R-module generated by df for $f \in R$ modulo:

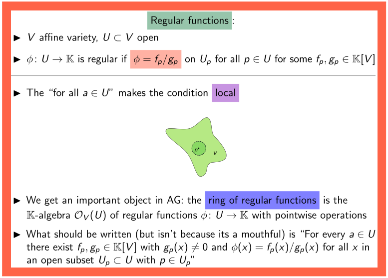
(i) $df = 0$ for $f \in \mathbb{K}$ (Scalars do not grow)

(ii) $d(f + g) = df + dg$ (Linear)

(iii) $d(fg) = f(dg) + (df)g$ (Product rule)

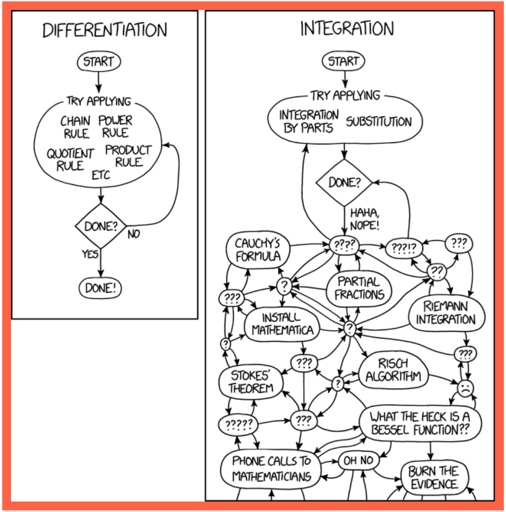
Kähler differentials

- In the localization the quotient rule is free, so this extends to regular functions



- As we will see, this is the key for using differentials in algebraic geometry

We have everything



- ▶ Differentiation "=" Product, chain, power and quotient rule
- ▶ XKCD ("=" experience) confirms that ☺

Thank you for your attention!

I hope that was of some help.