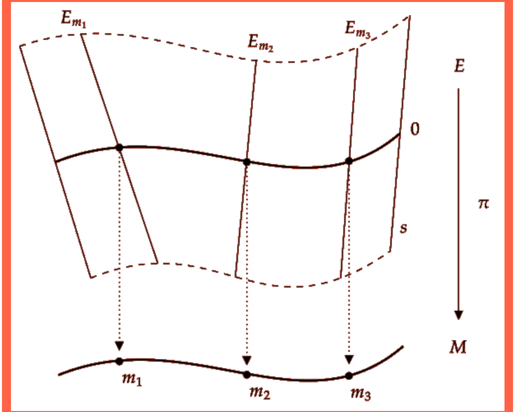
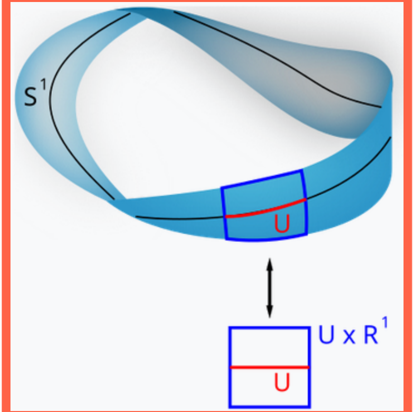
What are...Kähler differentials, part 2?

Or: Tangent bundles

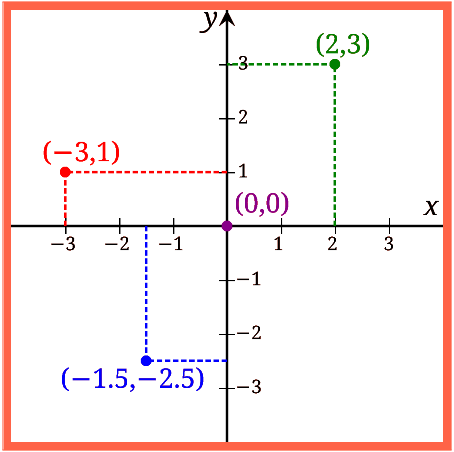
Tangent bundles





- **Vector bundle** = a linear algebra generalization of a cylinder
- **Idea** To every point of the space we attach a vector space so that these fit together to form another space of a desired type (manifold, or variety, or...)
- **Today** Tangent sheaves / bundles = tangent vector spaces at each point

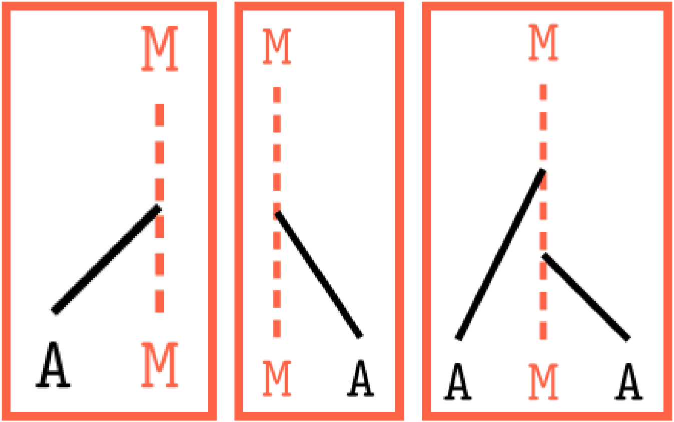
The affine plane



- Recall The coordinate ring of the affine plane is $R = \mathbb{C}[x, y]$
- Differentials It is easy to see that

$$\Omega_R = R dx \oplus R dy$$

Bimodules and multiplication



- Recall Bimodule = "a wall that you can hit with a ball from two sides"
- Example $R \otimes_{\mathbb{K}} R$ for an $\mathbb{K}$-module R
- Multiplication map $\delta\colon R \otimes_{\mathbb{K}} R \to R$ given by $f \otimes g \mapsto fg$

For completeness: A formal statement

For a $\mathbb{K}$-algebra R let Ω_R be as before, and $J = \ker \delta$

$$\Omega_R \cong J/J^2 \text{ as } R\text{-modules}$$

- Example For the affine plane, J is spanned by $1 \otimes x - x \otimes 1$ and $1 \otimes y - y \otimes 1$; and $dx \mapsto 1 \otimes x - x \otimes 1$ and $dy \mapsto 1 \otimes y - y \otimes 1$ is the isomorphism
- For a variety V, we can define a sheaf (cotangent bundle) Ω_V via pullback $i^*(\mathcal{J}/\mathcal{J}^2)$ (details omitted)

Lemma 14.8 (Ideal sequence). *Let $i\colon Y \to X$ be the inclusion of a closed subscheme.*

(a) *If $\mathscr{F}$ is a quasi-coherent sheaf on Y then $i_*\mathscr{F}$ is quasi-coherent on X.*

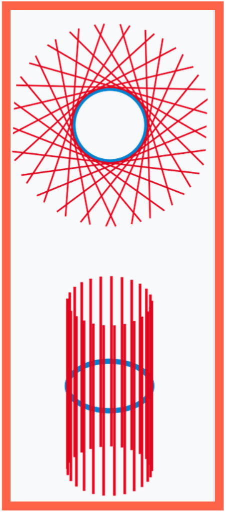
(b) *There is an exact sequence*

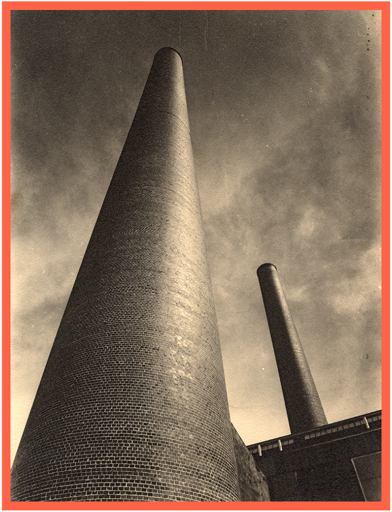
$$0 \to \mathscr{I}_{Y/X} \to \mathscr{O}_X \to i_*\mathscr{O}_Y \to 0$$

of quasi-coherent sheaves on X, where the second non-trivial map is the pull-back of regular functions as in Example 13.10 (a). Its kernel $\mathscr{I}_{Y/X}$ is called the ***ideal sheaf*** *of Y in X.*

For details and the picture see
https://agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf

Example





- Recall Projective space $\mathbb{P}^1$ is the circle S^1
- Example The tangent bundle on $\mathbb{P}^1$ is an infinite cylinder

Thank you for your attention!

I hope that was of some help.