What is...sheaf cohomology, part 1?

Or: Why do we want it?

Homology/cohomology

The torus T and the solid torus T^s

$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 2 \\ \dim H_2 = 1 \end{cases}$

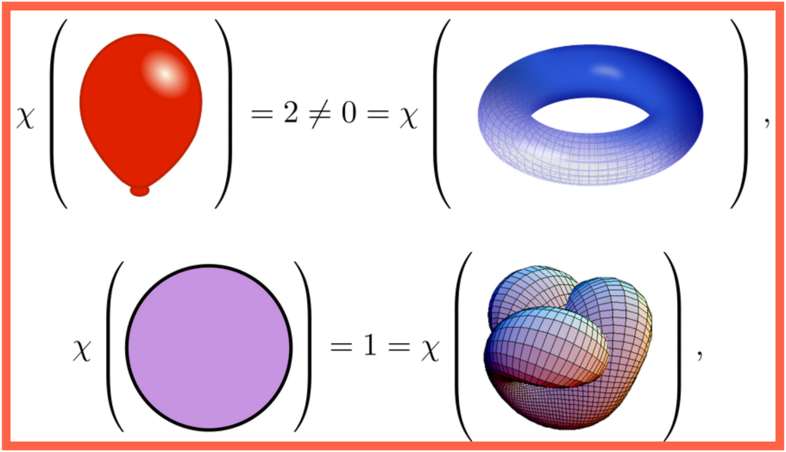
$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 1 \\ \dim H_2 = 0 \end{cases}$

- A zero dimensional hole $\dim H_0$ is a connected component
- A one dimensional hole $\dim H_1$ is the number of necklaces you can put it on
- A two dimensional hole $\dim H_2$ is the number of plugs needed to inflate it

Eric Weisstein "A hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point."

- Homology is one of the most important ideas in mathematics
- Why do topologists love homology?
- In this video I will not distinguish between homology and cohomology

It is an invariant!



- **Above** The Euler characteristic does not ‘detect’ the projective plane, but homology does
- **Idea** Two topological spaces are the same $\Rightarrow$ homology is the same; so: homology is different $\Rightarrow$ two topological spaces are different

Homology Computation

The code computes exclusively reduced homology of the given simplicial complexes.

```
Homology(X) : SmpCpx -> SeqEnum, SeqEnum

Homology(~X) : SmpCpx ->

Homology(X,A) : SmpCpx, Rng -> SeqEnum, SeqEnum

Homology(~X, A) : SmpCpx, Rng ->
```

- Homology = abelian groups = vector spaces with torsion
- Recall Vector spaces = linear algebra = powerful and computable
- Idea Homology = reduction of nonlinear things to linear things

For completeness: A formal statement

We will construct a cohomology theory:

(i) For quasi-coherent sheaves

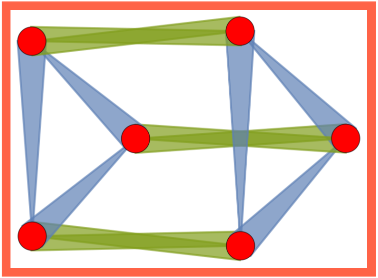
(ii) The construction uses, as usual, chain complexes

(iii) The result is an invariant (and can do other tricks!)

Sheaf cohomology

- The construction works more general, but I will ignore that
- There is a cohomology for sheaves of abelian groups on topological spaces

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$:



How good is it?

- Careful (Sheaf co)homology is actually not a great invariant (can you really reduce nonlinear things to a linear things?), but the problem is difficult
- Example Any pair of knots gives a homology 3-sphere by gluing their knot complements together switching meridian and longitude

Thank you for your attention!

I hope that was of some help.