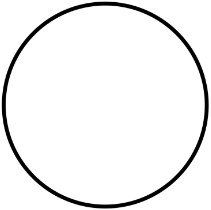
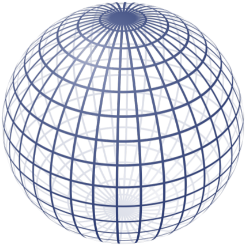
What is...sheaf cohomology, part 2?

Or: Properties and definition

The homology of the sphere



$\stackrel{?}{=}$



- H_* distinguishes spheres

$$H_n(S^d) \cong \begin{cases} \mathbb{Z} & n = 0, d \\ 0 & \text{else} \end{cases}$$

- Topology The homology of the sphere S^n is $1 + t^n$
- Algebraic geometry The homology of projective space $\mathbb{P}^n$ should be $1 + t^n$
- Truth The homology of $\bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$ on $\mathbb{P}^n$ is concentrated in degrees $0, n$

The homology of manifolds

The torus T and the solid torus T^s

$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 2 \\ \dim H_2 = 1 \end{cases}$

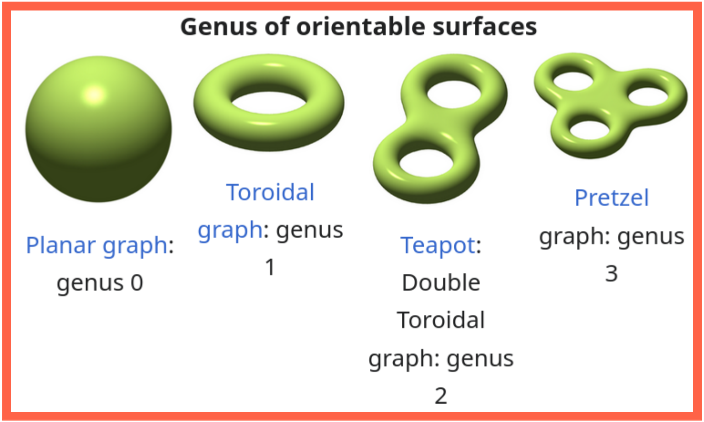
$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 1 \\ \dim H_2 = 0 \end{cases}$

- A zero dimensional hole $\dim H_0$ is a connected component
- A one dimensional hole $\dim H_1$ is the number of necklaces you can put it on
- A two dimensional hole $\dim H_2$ is the number of plugs needed to inflate it

Eric Weisstein "A hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point."

- Topology The homology of an $n$dim manifold is $a_0 + \ldots + a_n t^n$
- Algebraic geometry The homology of an $n$dim variety should be $a_0 + \ldots + a_n t^n$
- Truth This is true for projective and affine varieties

Genus = dimension of first homology group



- Topology The genus of an orientable surface is a_1
- Algebraic geometry The genus of a plane curve should be a_1
- Truth This is indeed true (using the structure sheaf)

For completeness: A formal statement

Cohomology of sheaves is defined as follows:

(i) Let $\mathcal{F}$ be a sheaf on V, and fix an affine open cover $U_1, \ldots, U_r$ (Setting)

(ii) $C^i = \bigoplus_{j_1 < \ldots < j_i} \mathcal{F}(U_{j_1} \cap \ldots \cap U_{j_i})$ (Cochain groups: $\mathbb{K}$ vector spaces)

(iii) $d^i : C^i \to C^{i+1}$ = sum over leaving out indexes; details skipped (Differential)

Cohomology: $ker d^i / im d^{i-1}$

- Theorem This is well-defined ($d^2 = 0$) and independent of the cover

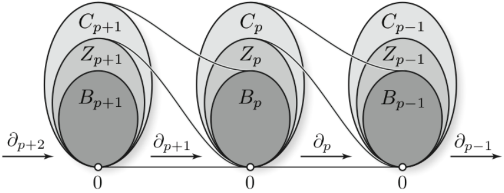
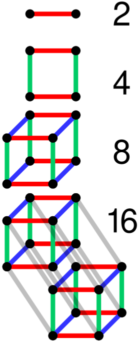
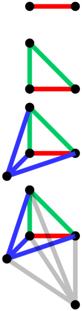


Image source: S. Balakrishnan et al. Minimax Rates for Homology Inference.

- Cohomology is an invariant of sheaves

Euler characteristic





n	V	E	F	C	χ	V	E	F	C	χ
	+	−	+	−	=	+	−	+	−	=
1	2	–	–	–	**2**	2	–	–	–	**2**
2	4	4	–	–	**0**	3	3	–	–	**0**
3	8	12	6	–	**2**	4	6	4	–	**2**
4	16	32	24	8	**0**	5	10	10	5	**0**

- **Topology** The Euler characteristic is $a_0 - a_1 \pm \ldots$
- **Algebraic geometry** The Euler characteristic should be $a_0 - a_1 \pm \ldots$
- **Truth** This is indeed true (and additive on exact sequences)

Thank you for your attention!

I hope that was of some help.