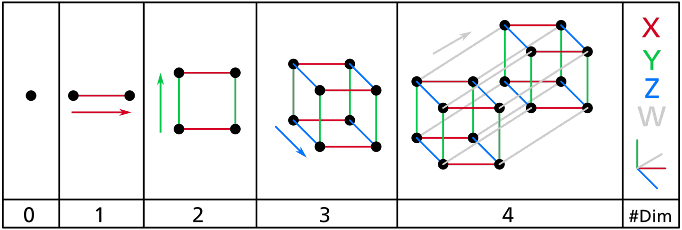
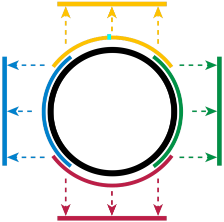
What is...the dimension of a variety?

Or: A space in a space in a space...

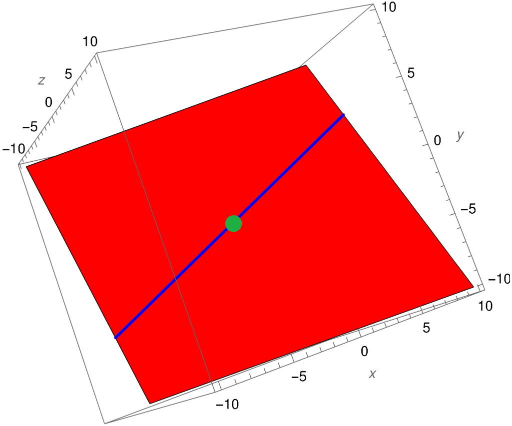
Dimension in general





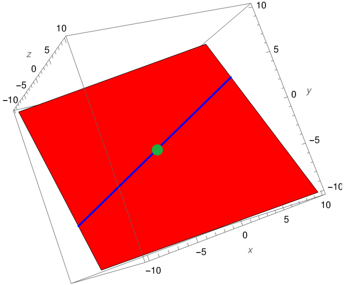
- Dimension = measurement of "size" of a thing
- The precise definition varies depending on the problem
- Today Dimension for varieties (over arbitrary fields)

Nested spaces



- Nesting strategy = Put a point in a line in a plane in ...
- The space $\mathbb{R}^3$ is then 3d since it contains maximally three smaller objects
- Why is that good? Makes intuitively sense and is field independent

An algebra version



- Krull dimension = maximal length of chains of prime ideals in an algebra A:

$$I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_n = A$$

- Example The above is the variety version of the coordinate ring inclusions:

$$(0) \subset (x) \subset (x, y) \subset (x, y, z) = \mathbb{R}[x, y, z]$$

- This is very similar to the notion from the previous slide

For completeness: A formal statement

On the variety side for $V \neq \emptyset$:

- Consider chains of closed irreducible subvarieties $\neq \emptyset$:

$$V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$$

- Then $\dim V = \sup\{\text{length of all such chains}\} \in \mathbb{N} \cup \{\infty\}$

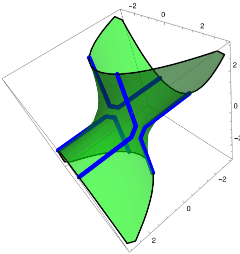
On the algebra side for $A \neq \emptyset$:

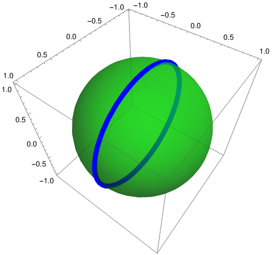
- Consider chains of prime ideals $\neq \emptyset$:

$$I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_n = A$$

- Then $\dim A = \sup\{\text{length of all such chains}\} \in \mathbb{N} \cup \{\infty\}$

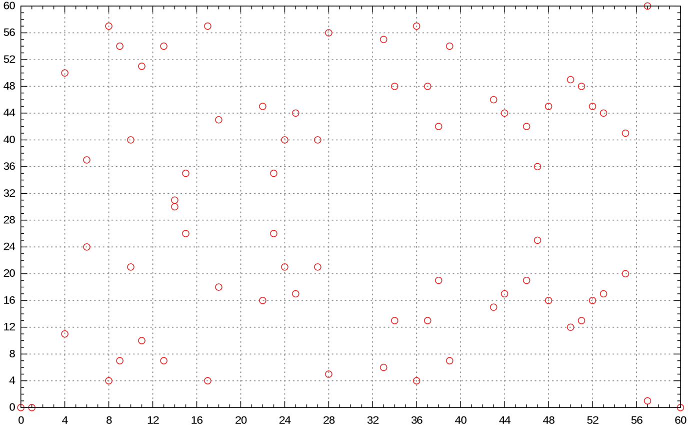
These agree $\dim V = \dim \mathbb{K}[V]$





Pure dimensional spaces

$y^2 = x^3 - x$ over $\mathbb{F}_{61}$:



- V is of pure dimension n if every irreducible component is of dimension n
- Examples Curves are pure 1d, surfaces are pure 2d
- Careful This depends on $\mathbb{K}$, e.g. a complex curve is of real dimension two

Thank you for your attention!

I hope that was of some help.