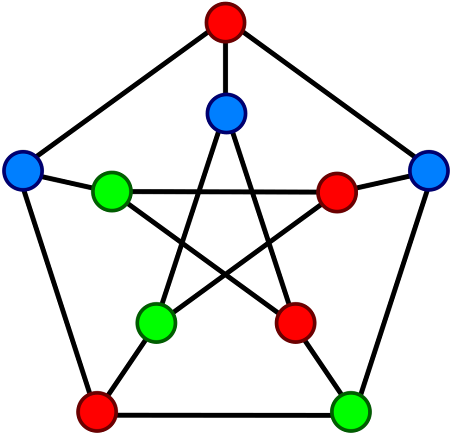
What is...the chromatic polynomial?

Or: Polynomials and colors

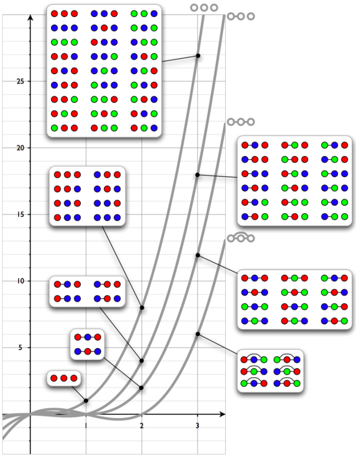
Counting colorings

3-coloring:



- Coloring = vertices get colors such that two adjacent vertices have different colors
- Let $P_G(k)$=number of k-colorings of G (colorings using k colors)
- Question How are the $P_G(k)$ related?

3 vertex graph



edgeless: x^3

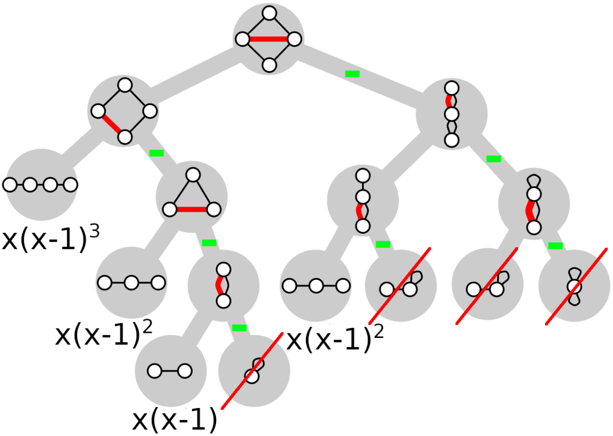
one edge: $x^2(x-1)$

line: $x(x-1)^2$

circle: $x(x-1)(x-2)$

- For $G = \bullet\bullet\bullet$ we have a polynomial giving $P_G(k)$, namely $P_G(x) = x^3$
- For other small graphs one checks that the same works
- Question Is there a polynomial counting colorings?

Deletion-contraction



$$\Rightarrow P_G(x) = \begin{aligned} &+ x(x-1)^3 \\ &- 2x(x-1)^2 \\ &+ x(x-1) \end{aligned}$$

- Here is an algorithm to compute $P_G(x)$
- Starting condition $P_{tree}(x) = x(x-1)^{\#vertices-1}$ and $P_{loop}(x) = 0$
- Then use deletion-contraction: $P_G(x) = P_{G\setminus e}(x) - P_{G/e}(x)$

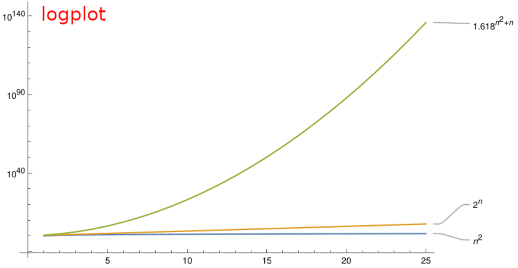
For completeness: A formal statement

There exists a polynomial $P_G(x)$ associated to a graph such that:

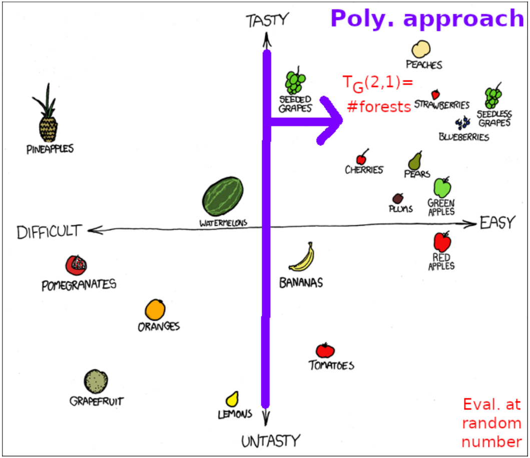
$$P_G(k) = \#k\text{-colorings}$$

- The polynomial is called chromatic polynomial
- The polynomial can be computed by deletion-contraction
- However, the runtime is quite bad: $\approx \phi^{\#vertices+\#edges}$; ϕ=golden ratio

$\#edges \approx \#vertices^2$:



Whatever “easy” means



- Recall that graph polynomials are for easy problems
- The runtime for $P_G(x)$ is horrible, so how can that be easy?
- This is easy in the sense that we get all colorings at once

Thank you for your attention!

I hope that was of some help.