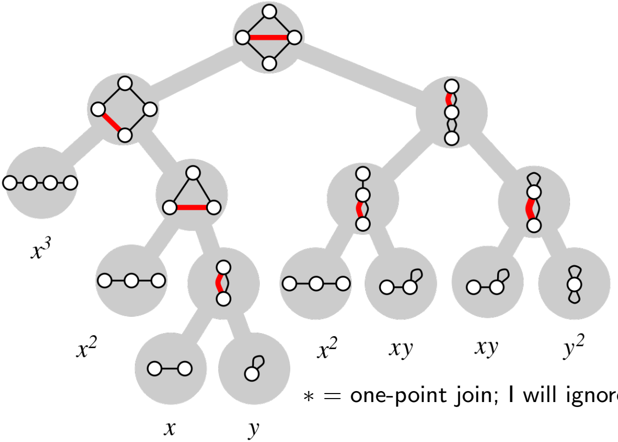
What is...Tutte universality?

Or: Everyone discovers the same polynomial

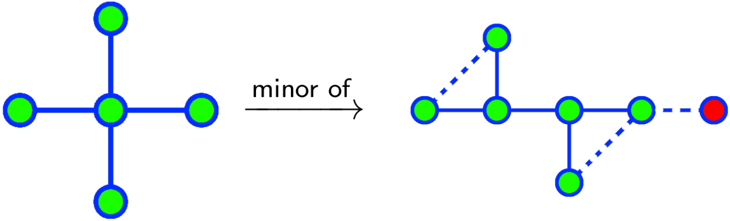
The Tutte polynomial $T_G(x, y)$ – reminder

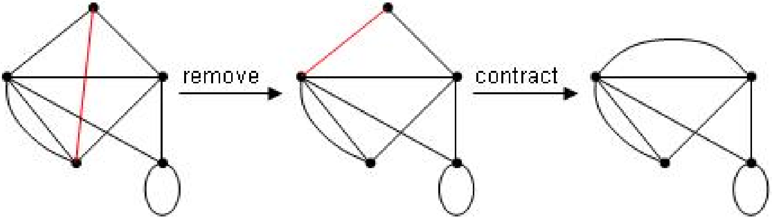


$*$ = one-point join; I will ignore it

- Recall deletion-contraction: $T_G(x, y) = T_{G\backslash e}(x, y) + T_{G/e}(x, y)$
- We have $T_{G \cup H}(x, y) = T_G(x, y) \cdot T_H(x, y)$ and $T_{G*H}(x, y) = T_G(x, y) \cdot T_H(x, y)$
- How unique is the Tutte polynomial wrt these properties?

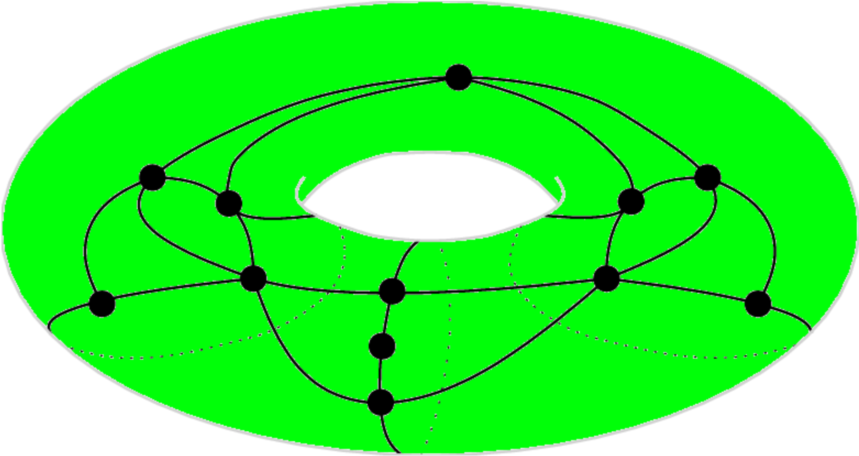
Graph minor





- Minor = can be formed by deleting vertices and edges and contracting edges
- We are interested in minor closed classes *i.e.* $(G \in \mathcal{C}) \Rightarrow (\text{minors of } G \in \mathcal{C})$

Minor closed



- Example $\mathcal{C}$ = all finite graphs is minor closed
- Example $\mathcal{C}$ = all planar graphs is minor closed
- Example $\mathcal{C}$ = all graphs embedded into a fixed surface is minor closed

For completeness: A formal statement

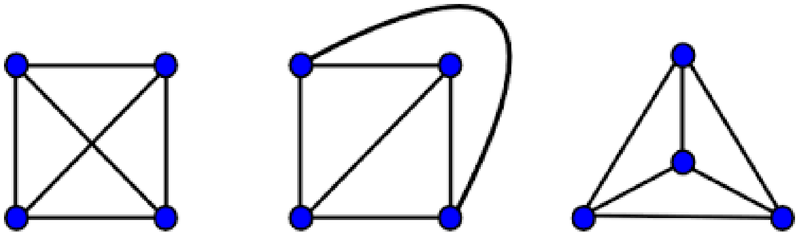
The Tutte polynomial is universal, formally:

Assume that $f:\mathcal{C}\to\mathbb{C}[x,y]$ for $\mathcal{C}$ minor closed satisfies:

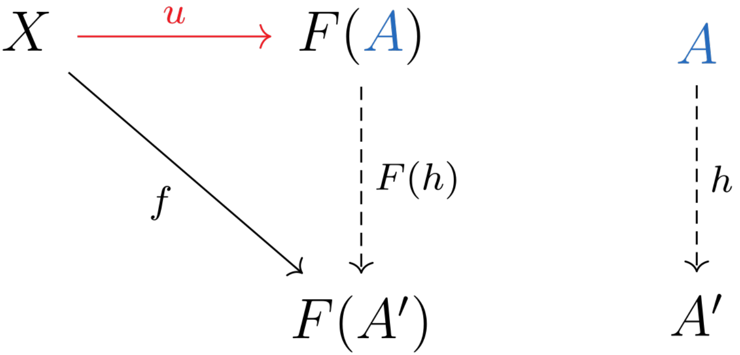
- $f(G)=af(G\setminus e)+bf(G/e)$ Deletion-contraction
- $f(G\cup H)=f(G)\cdot f(H)$ and $f(G*H)=f(G)\cdot f(H)$ Product
- $f(\bullet)=1$ Normalization

Then f is a specialization of the Tutte polynomial, up to scaling

- For minor closed we can always take all finite graphs
- The above actually says that $T_G(x,y)$ is already detected on *e.g.* planar graphs



All polynomial are specializations of Tutte



- Consequence "All" graph polynomials are specializations of $T_G(x, y)$
- $T_G(x, y)$ is a universal graph invariant
- In other words, you could have discovered it!

Thank you for your attention!

I hope that was of some help.