What is...the complexity of the Tutte polynomial?

Or: Easy or difficult?

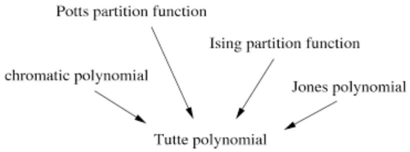
Computing the Tutte polynomial $T_G(x, y)$

For completeness: A formal statement

There exists a polynomial $T_G(x, y)$ associated to a graph such that:

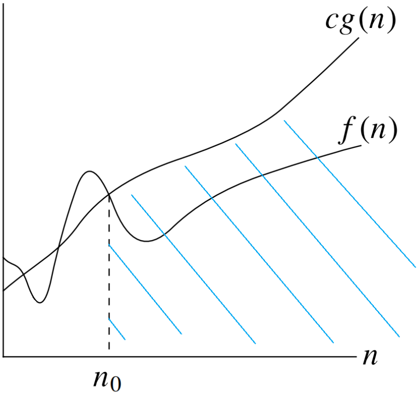
- $T_G(2, 1) = \#$ forests
- $T_G(1, 1) = \#$ spanning forests
- $T_G(1, 2) = \#$ spanning subgraphs
- More...

- The polynomial is called Tutte polynomial
- Also we have the specialization "chromatic(x) = Tutte(x,0)", and more



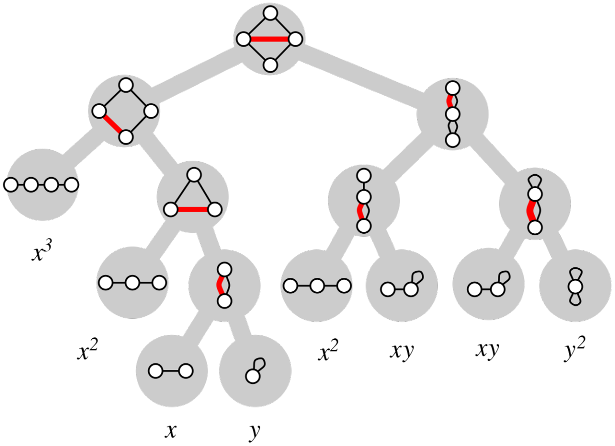
- $T_G(x, y)$ counts many things, so it would be good to compute it efficiently
- Question How difficult is it to compute $T_G(x, y)$?
- Question How difficult is it to compute $T_G(a, b)$ for $(a, b) \in \mathbb{C}^2$?

Landau–Bachmann notation



- We say $f \in O(g)$ if $f(n) \leq cg(n)$, c=constant, from some point onward
- Example $10000n \in O(n^2)$
- We use this to analyze worst-case runtime for algorithms

The computation via recursion

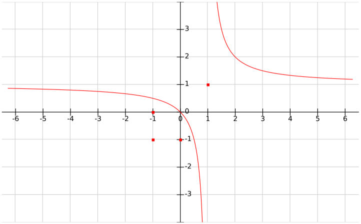


- Recall the deletion-contraction way to compute $T_G(x, y)$
- This looks like exponential growth
- Guess Computing $T_G(x, y)$ is in probably difficult *e.g.* *Tutte* $\in O(2^{\#edges})$

For completeness: A formal statement

The computation of $T_G(a, b)$ is...

- ...in $O(polynomial)$ for $(a-1)(b-1) = 1$ Easy



- ...in $O(polynomial)$ for ($j = \exp(2\pi i/3)$)
  $(a, b) \in \{(1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2), (j^2, j)\}$ Easy
- ...#P hard otherwise Hard

- #P hard $\approx$ *Tutte* $\in O(2^{\#edges})$ but the precise runtime is unknown
- Note the huge difference between general and specific points

Difficult in general, but...

Graph class	♯**P**-hard	subexponential	**FPT**	**P**
All graphs	$\mathbb{C}^2 - H$	H	H	H
planar	$\mathbb{C}^2 - H_2$	H_2	H_2	H_2
bipartite planar	$\mathbb{C}^2 - H_{b-p}$	H_{b-p}	H_{b-p}	H_{b-p}
$TW(k)$	$\emptyset$	$\mathbb{C}^2$	$\mathbb{C}^2$	H
$CW(k)$	$\emptyset$	$\mathbb{C}^2$	H	H

H = hyperbola from the previous slide

$TW(k)$ = tree width at most k

$CW(k)$ = clique width at most k

- Computing $T_G(x, y)$ in general is difficult
- Computing $T_G(x, y)$ in special cases is not so bad

Thank you for your attention!

I hope that was of some help.