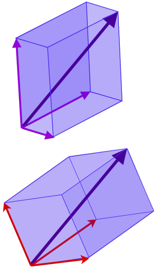
What is a...matroid?

Or: Bases, forests, partitions and friends

Fundamental properties of bases

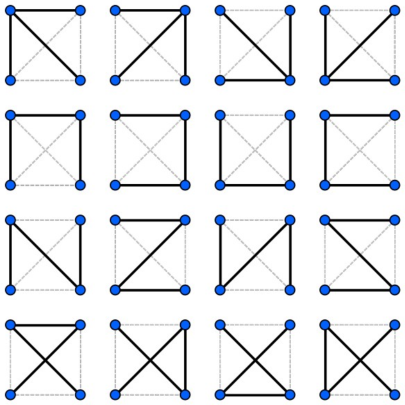


$$, \quad \mathbf{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- **Question** What makes bases of vector spaces special?
- **Answer attempt 1** They exist!
- **Answer attempt 2** We can always exchange vectors between them without loosing the property of being a basis

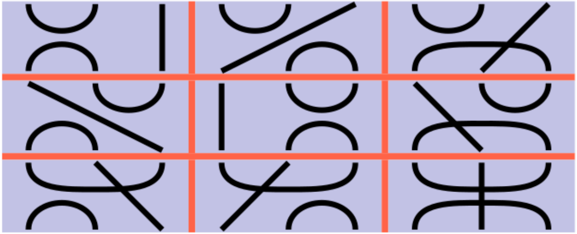
Fundamental properties of forests

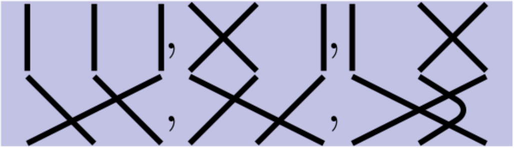


- Question What makes spanning forests of graphs special?
- Answer attempt 1 They exist!
- Answer attempt 2 We can always exchange edges between them without loosing the property of being a spanning forest

Fundamental properties of partitions

2-partitions of $\{1, 2, 3, 1', 2', 3'\}$:





- Question What makes 2-partitions of a set special?
- Answer attempt 1 They exist!
- Answer attempt 2 We can always exchange elements between them without loosing the property of being a 2-partition

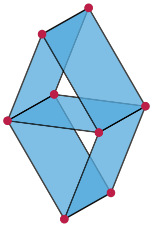
For completeness: A formal definition

A matroid is a pair $(E, \mathfrak{B})$ of a finite set E and bases $\mathfrak{B} \subset \mathfrak{P}(E)$ such that:

(i) $\mathfrak{B}$ is not empty Existence of bases

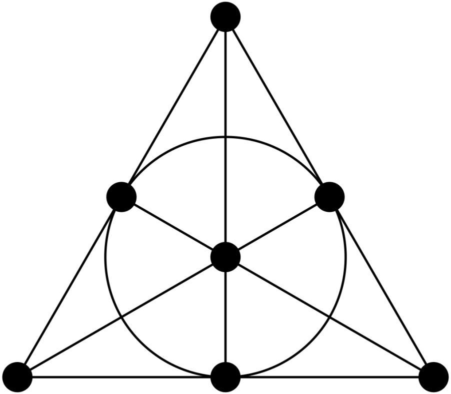
(ii) For $A \neq B$ in $\mathfrak{B}$ and $a \in A \setminus B$ there exists $b \in B$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathfrak{B}$ Basis exchange property

- Examples include the ones on the previous three slides
- Take eight points and bases = collection of four points which are not the ones in the picture



This is a slightly obscure matroid

Matroids are everywhere



- Matroids (vastly) generalize bases, forests and partitions
- Example Fano matroid with seven points and bases being the lines above
- Strictly speaking this example is linear, but that is not immediate

Thank you for your attention!

I hope that was of some help.