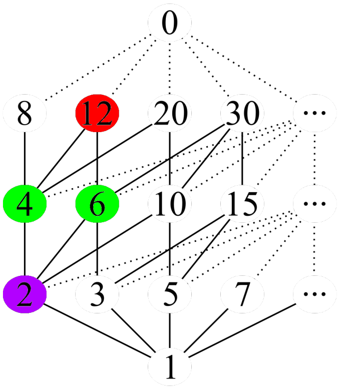
What is...the lattice of flats?

Or: The properties of inclusion

Lattices generalize division and inclusion



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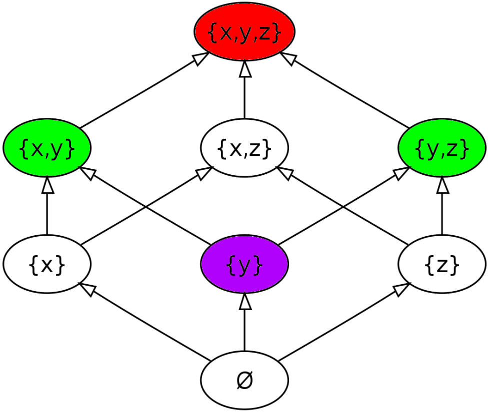
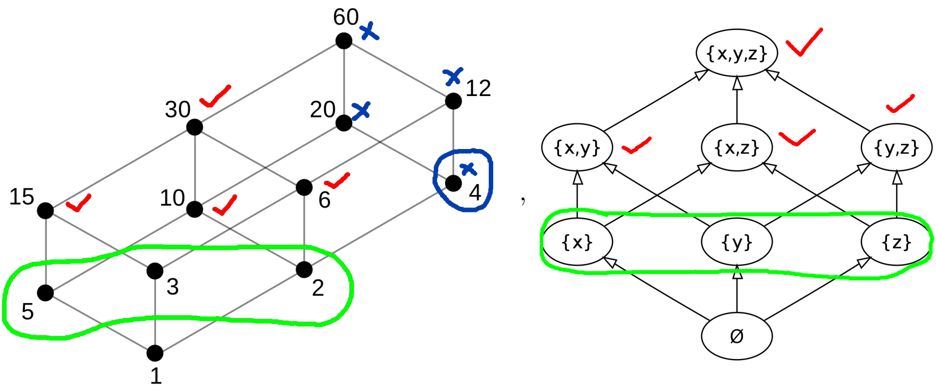


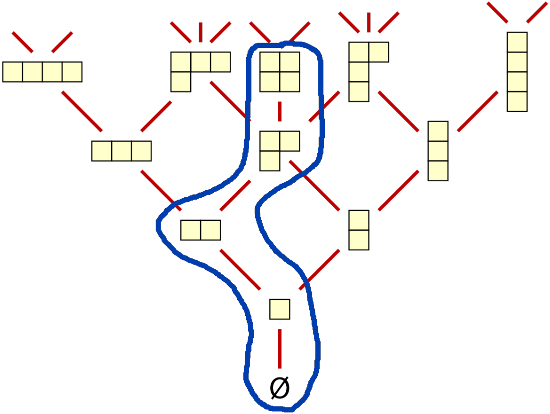
- **Lattice** = poset such that every pair (x, y) has a join $x \vee y$ and a meet $x \wedge y$
- **Example** $\mathbb{Z}_{\geq 0}$ with division, join=lcm, meet=gcd
- **Example** Sets with inclusion, join=$\cup$, meet=$\cap$

Atoms of a lattice



- (Co)Atoms of a lattice = Covers of the least (biggest) element
- Atomic = every element can be written as the join of atoms
- Examples Division lattices are mostly not atomic, inclusion lattice often are

The rank again



- Rank = length of a saturated chain
- Saturated = successive elements cover one another
- Example In the Young lattice the rank is the number of boxes

For completeness: A formal statement

Flats of a matroid form a geometric lattice

- Here is the definition of a matroid using flats:

Theorem 2.52. *Let E be a finite set and let $\mathcal{F}$ be a family of subsets of E. Then the family $\mathcal{F}$ are the flats of a matroid if and only if:*

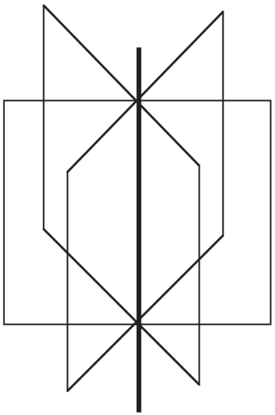
(F1) $E \in \mathcal{F}$.

(F2) *If* $F_1, F_2 \in \mathcal{F}$, *then* $F_1 \cap F_2 \in \mathcal{F}$.

(F3) *If* $F \in \mathcal{F}$ *and* $\{F_1, F_2, \ldots, F_k\}$ *is the set of flats that cover* F, *then* $\{F_1 - F, F_2 - F, \ldots, F_k - F\}$ *partition* $E - F$.

- Geometric lattice $=$ lattice $+$ atomic $+$ $(rk(x \vee y) + rk(x \wedge y) \leq rk(x) + rk(y))$
- Example In a linear matroid flats are linear subspaces (lines, planes, *etc.*)

The axiom (F3)



- Geometric motivation for (F3) = given a line in $\mathbb{R}^3$, the planes that contain this line partition the rest of $\mathbb{R}^3$
- There are infinitely many such planes, but its a pain to illustrate that many planes ;-)

Thank you for your attention!

I hope that was of some help.