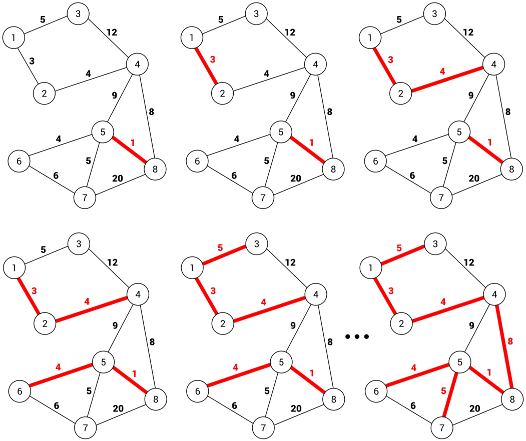
What are...matroid embeddings?

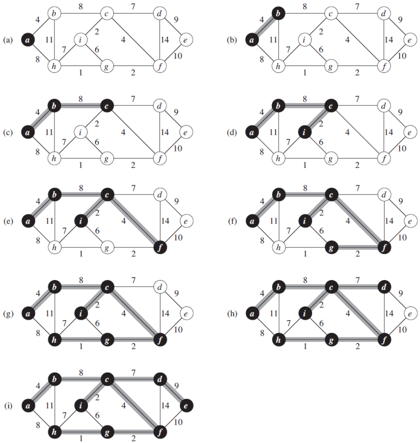
Or: This actually only about greedoids...

Greedy strategy for spanning forests – à la Kruskal



- Spanning forests can be found using a greedy strategy
- Algorithm Order the edges by weight and take the minimal admissible edge
- This is realized by a matroid

Greedy strategy for spanning forests – à la Prim



- Spanning forests can be found using a greedy strategy
- Algorithm Grow the tree from a vertex by taking the minimal admissible edge
- This is not realized by a matroid

Greedoids include Prim

A matroid is a pair $(E, \mathfrak{I})$ of a finite set E and LI sets $\mathfrak{I} \subset \mathfrak{P}(E)$ such that:

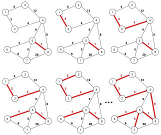
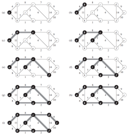
(i) $\mathfrak{I}$ is not empty Existence of LI sets

(ii) For $I \subset J$ and $J \in \mathfrak{I}$ implies $I \in \mathfrak{I}$, and for $|I| < |J|$ there exists $i \in J \setminus I$ such that $I \cup \{i\} \in \mathfrak{I}$ Vector exchange property

A greedoid is a pair $(E, \mathfrak{F})$ of a finite set E and feasible (F) sets $\mathfrak{F} \subset \mathfrak{P}(E)$ such that:

(i) Every $I \in \mathfrak{F}, I \neq \emptyset$ contains i such that $I \setminus \{i\} \in \mathfrak{F}$ Existence of F sets

(ii) For $I, J \in \mathfrak{F}$ with $|I| < |J|$ there exists $i \in J \setminus I$ such that $I \cup \{i\} \in \mathfrak{F}$ Vector exchange property

included:  $+$ 

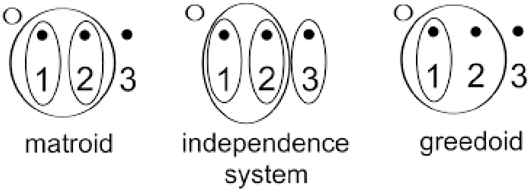
- Greedoid $=$ mild generalization of a matroid
- Greedoids admit greedy strategies
- Examples Every matroid is a greedoid, but also $\exists$ Prim's greedoid

For completeness: A formal statement

We have the following:

- The greedy algorithm works for all greedoids and all admissible weightings (this works similarly as for matroids)
- The converse is almost true as well

- Thus, matroids/greedoids ≈ perfect greed



- However, greedoids are both, too general and too constraining:
  - ▷ The greedy algorithm need not return an optimal solution on a greedoid;
  - ▷ The are greedy strategies not coming from a greedoid
- The "correct" notion is that of a matroid embedding

Matroid embeddings – finally

In combinatorics, a **matroid embedding** is a set system $(\boldsymbol{F}, E)$, where $\boldsymbol{F}$ is a collection of *feasible sets*, that satisfies the following properties.

1. Accessibility property: Every non-empty feasible set X contains an element x such that $X \setminus \{x\}$ is feasible.
2. Extensibility property: For every feasible subset X of a *basis* (i.e., maximal feasible set) B, some element in B but not in X belongs to the **extension** ext(X) of X, where ext(X) is the set of all elements e not in X such that $X \cup \{e\}$ is feasible.
3. Closure–congruence property: For every superset A of a feasible set X disjoint from ext(X), $A \cup \{e\}$ is contained in some feasible set for either all e or no e in ext(X).
4. The collection of all subsets of feasible sets forms a matroid.

Matroid embedding was introduced by Helman, Moret & Shapiro (1993) to characterize problems that can be optimized by a greedy algorithm.

- Matroid embedding = whatever you see above
- The point All greedy situations come from these matroid embeddings
- This comes up by answering: "If I have a greedy strategy, how can I cook up a matroid?"

Thank you for your attention!

I hope that was of some help.