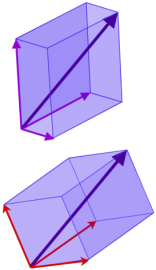
What are...representable matroids?

Or: Matrices and matroids

Linear matroids



$$, \quad \mathbf{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$B_1 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\1 \end{bmatrix} \right\}$$

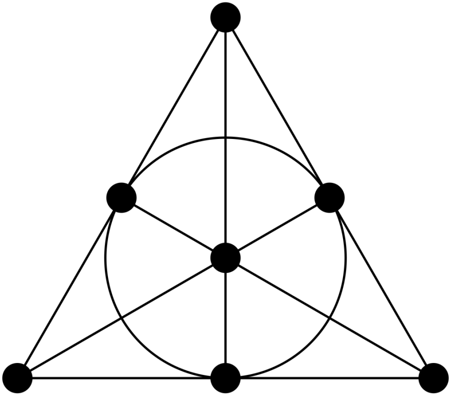
- Linear matroid = obtained from a matrix by taking sets of linearly independent column vectors
- Every matrix gives rise to a matroid in this way
- What about the converse ?

A subprogram of linear algebra?

Matroid object	Linear algebra theorem
Bases	Every basis of a finite-dimensional vector space has the same size.
Independent sets	Every linearly independent set can be extended to a basis.
Flats	The intersection of subspaces is a subspace.
Flats	The subspaces that cover a given subspace W partition $V - W$.
Rank function	If U and W are subspaces, then $\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U + W)$.

- Representable matroid = a matroid coming from a linear matroid
- Problem A matroid could come from a linear one in some obscure way
- If all matroids a representable, then matroid theory $\subset$ linear algebra

The greedy strategy

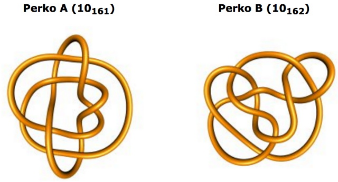


- Problem Combinatorial matroids can be representable in a nontrivial way
- Recall Fano matroid with seven points and the bases being the sets of three points that are not illustrated
- Example The Fano matroid is representable $\Leftrightarrow \mathrm{char}(\text{field}) = 2$

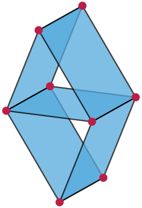
For completeness: A formal statement

Non-representable matroids exist

- It took some time to find an example
  Proving that something does not work is difficult

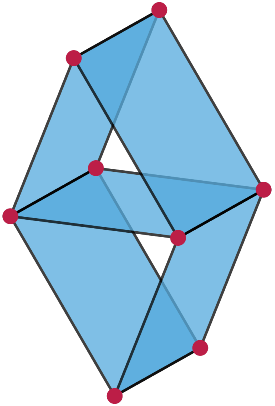


- The smallest non-representable matroid is on eight elements



Vámos again

the bases are
all non-displayed:
ones of size four



- The Vámos matroid is not representable over any field
- "Proof": Assume otherwise, collect equations and show no solution exists
- In general, proving non-representability is a bit painful but more another time

Thank you for your attention!

I hope that was of some help.