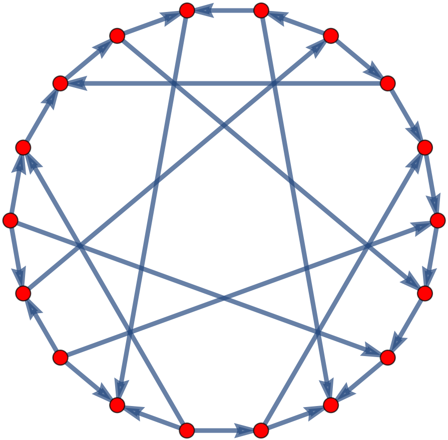
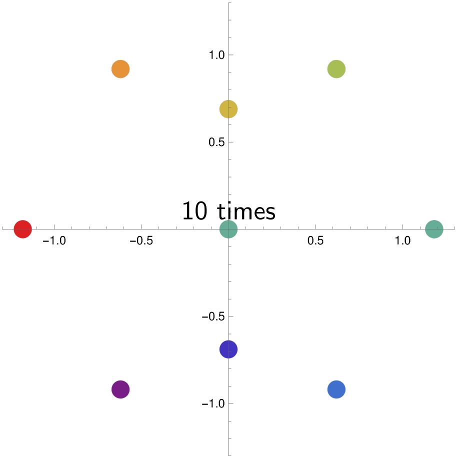
What is...path counting via eigenvalues?

Or: Paths “=” matrix powers “=” largest eigenvalue

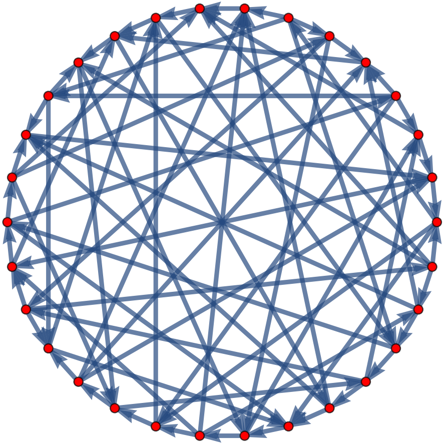
Spectrum of a(n oriented multi)graph

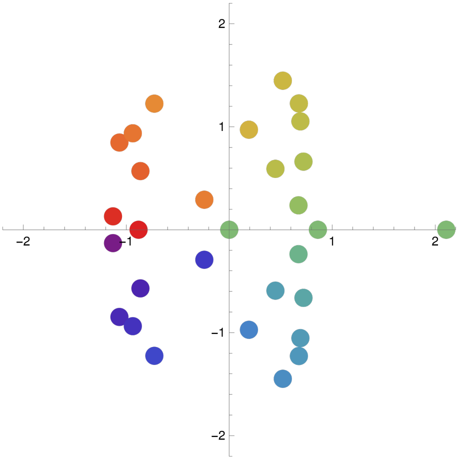




- Spectrum of a matrix "=" the (multi)set of eigenvalues in $\mathbb{C} = \mathbb{R}^2$
- Spectrum $S(G)$ of G "=" spectrum of $A(G)$
- Question What does $S(G)$ know about G?

Spectral patterns






```
{3.50743, 1.34522, 2.71702, 1.3628, 2.15418, 0.724802, 0.51078, 1.41901, 0.59805, 0.94177, 0.660077, 0.481037, 1.29229, 0.938406,
 0.992614, 0.696089, 1.34923, 1.03124, 0.896952, 0.81209, 0.763617, 0.305472, 1.28443, 0., 0.812626, 1.02427, 1.08017, 1.77954, 0.751953, 1.}
```

- ▶ Leading eigenvalue There is a real biggest eigenvalue $PF(G)$
- ▶ $PF(G)$ has an associated nonnegative eigenvector

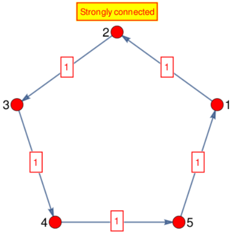
The Perron–Frobenius theorem

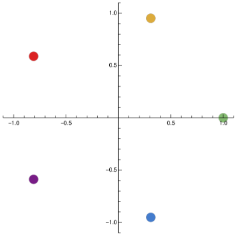
Let M be an irreducible matrix with entries in $\mathbb{N}_0$. Then:

(a) There exists a unique eigenvalue $pf \in \mathbb{R}_{>0}$ of M whose absolute value is bigger than those of other eigenvalues The leading eigenvalue

(b) Up to scalars, there is a unique eigenvector PF with entries from $\mathbb{R}_{>0}$, and it has eigenvalue pf The leading eigenvector

(c) The only eigenvectors with the same absolute value as pf are on the same circle as pf Symmetry of the eigenvalues





- Irreducible "=" G is connected (otherwise run per connected component)
- Let us see what this implies for path counting!

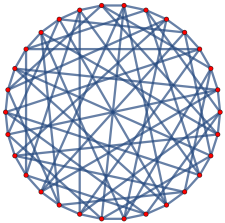
For completeness: A formal statement

For any (oriented multi)graph G:

- ▷ We have a leading eigenvalue $PF(G)$ and associated leading left v and right w eigenvectors with $w^T v = 1$
- ▷ We have $\lim_{n\to\infty} A(G)^n/PF(G)^n = vw^T$

Thus, the growth rate of the number of paths is controlled by $PF(G)$, v and w

- Upshot Linear algebra methods help to count paths!
- We only need to know data associated to $A(G)$ to "count" the number of paths of arbitrary length

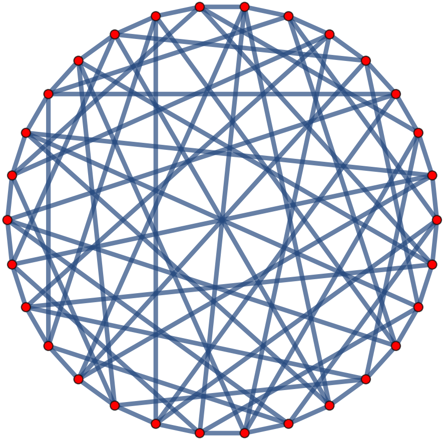
$G =$ 

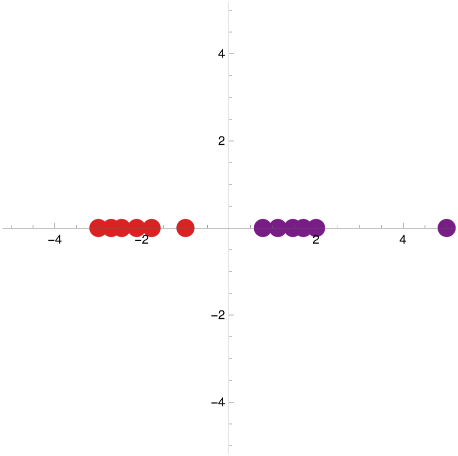
$$PF(G) = 5$$

$$A(G)^{100}/5^{100} = 1/30 \cdot \text{(only 1 matrix)}$$

$$v = w = (1, ..., 1), w^T v = 1/30$$

The graph case is even a bit better





- $\boxed{A(G) \text{ is symmetric}}$ for a graph G
- Thus, all eigenvalues are real
- Thus, $v = w$ (up to scaling)

Thank you for your attention!

I hope that was of some help.