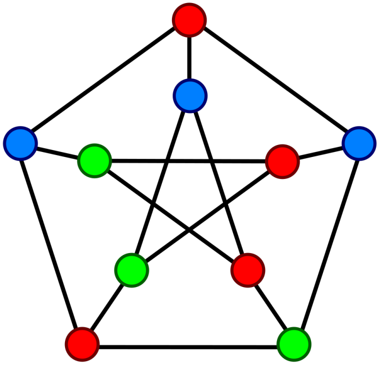
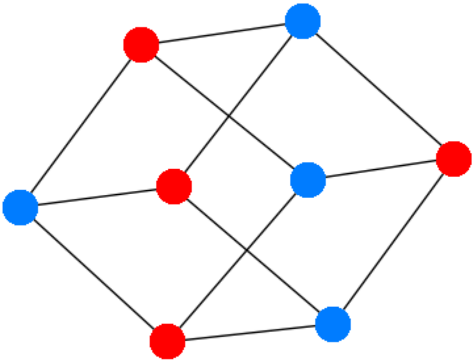
What are...applications of the spectrum?

Or: The spectrum knows a lot!

Characterizing graphs



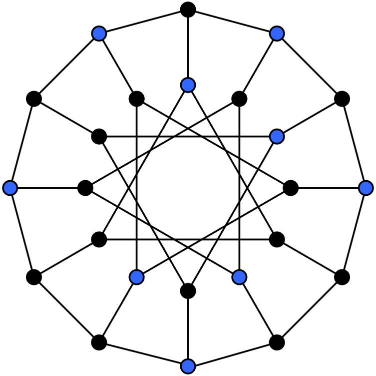
$S = \{3, 1, 1, 1, 1, 1, -2, -2, -2, -2\}$



$S = \{3, 1, 1, 1, -1, -1, -1, -3\}$

- The spectrum $S = \{\lambda_1 \geq ... \geq \lambda_n\}$ can characterize graphs
- Example G is k regular $\Leftrightarrow \lambda_1^2 + ... + \lambda_n^2 = kn$
- Example G is bipartite $\Leftrightarrow (\lambda_i \in S \Rightarrow -\lambda_i \in S$ with same multiplicity)

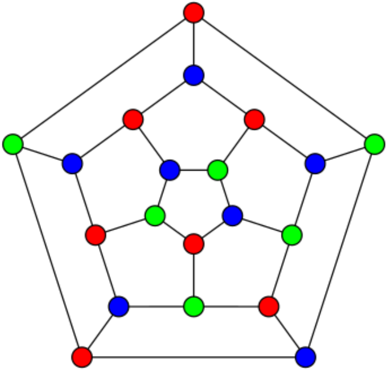
Detecting clusters



$S = \{3, 2^6, 1^3, 0^4, -1^3, -2^6, -3\}$, $8 \leq -24 \cdot 3 \cdot (-3)/(3^2 - 3 \cdot (-3)) = 12$

- **Coclique** = a set of pairwise nonadjacent vertices
- **Independence number** $\alpha(G)$ = size of the largest coclique
- **Example** We have $\alpha(G) \leq -n\lambda_1\lambda_n/(\delta^2 - \lambda_1\lambda_n)$; δ = minimum vertex degree

Knowing colorings



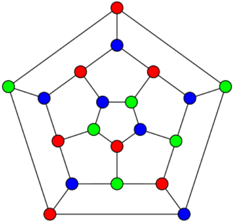
$S = \{3, \sqrt{5}^3, 1^5, 0^4, -2^4, -\sqrt{5}^3\}$, $2.34 \leq 3 \leq \chi(G) \leq 4$

- The spectrum knows colorings, e.g. the chromatic number $\chi(G)$
- Example If G is connected, then $\chi(G) \leq 1 + \lambda_1$
- Example If G is not edgeless, then $\chi(G) \geq 1 - \lambda_1/\lambda_n$

For completeness: A formal statement

The graph spectrum has many applications, e.g.:

- For characterizing graphs My first example
- For (co)cliques My second example
- For colorings My third example
- For variations, e.g. $\chi(G) \geq \min(1 + \mathit{mult}(\lambda_n), 1 - \lambda_n/\lambda_2)$ for $\lambda_2 > 0$
- For many more, e.g. Shannon capacity (we will see this maybe later)



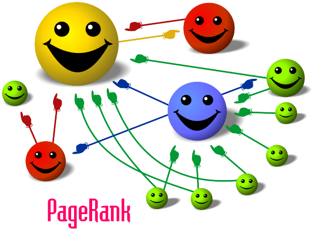
$$S = \{3, \sqrt{5}^3, 1^5, 0^4, -2^4, -\sqrt{5}^3\}$$

PageRank

Suppose pages x_1, ..., x_m are the pages that link to a page y. Let page x_i have d_i outgoing links. Then the PageRank of y is given by

$$PR(y) = 1 - \alpha + \alpha \sum_i \frac{PR(x_i)}{d_i}.$$

The PageRanks form a probability distribution: $\sum_x PR(x) = 1$. The vector of PageRanks can be calculated using a simple iterative algorithm, and corresponds to the principal eigenvector of the normalized link matrix of the web. A PageRank for 26 million web pages can be computed in a few hours on a medium size workstation. A suitable value for α is $\alpha = 0.85$.



- Google's PageRank is one of the most crucial applications of the spectrum
- There will be a whole video explaining it

Thank you for your attention!

I hope that was of some help.