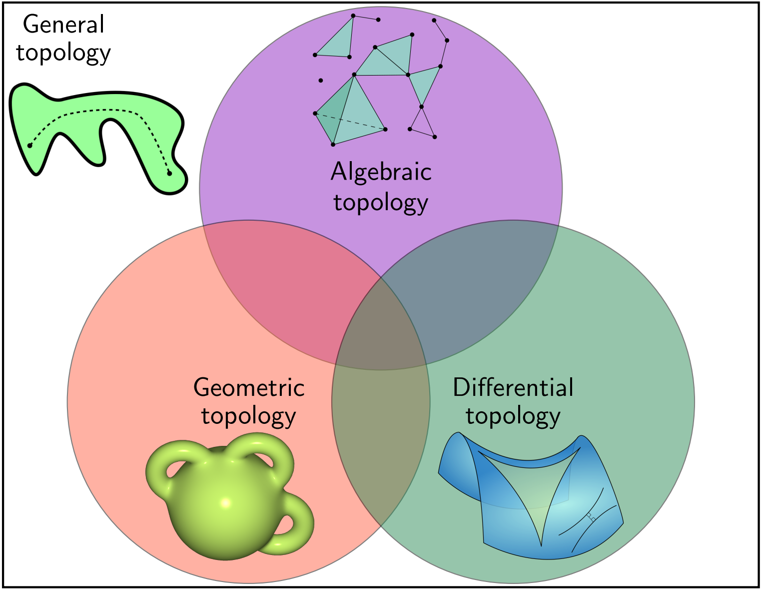
What is...algebraic topology?

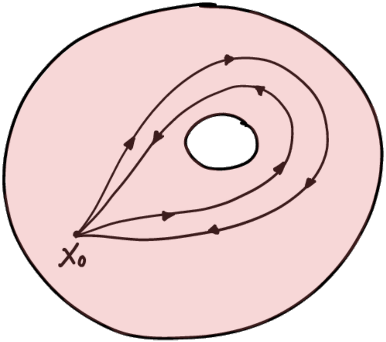
Or: Translating from topology to algebra

The main fields of topology



Algebraic topology translates topology/geometry into algebra & combinatorics

Topology to combinatorics – the fundamental group $\pi_1(X)$



Loops up to deformation in a topological space X form a group $\pi_1(X)$:

- Concatenation gives composition Multiplication
- Concatenation is associative up to rescaling Associativity
- "I do not move" is a unit Unit
- "Move backwards" is the inverse Inverse

This is a combinatorial invariant

The keywords – what (a classical course in) algebraic topology studies

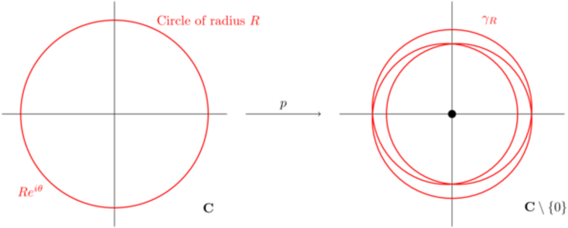
- Homotopy a.k.a. easy-to-explain-hard-to-compute
  - Fundamental group
  - Seifert–van Kampen theorem
  - Homotopy groups
  - ...
- Homology & cohomology a.k.a. easy-to-compute-hard-to-explain
  - Simplicial & singular homology
  - Mayer–Vietoris sequence
  - Hurewicz theorem
  - ...
- One also sees generalizations
  - Categorical aspects
  - Homological algebra
  - Homotopical algebra
  - ...

Application one – the fundamental theorem of algebra

Question: Do all non-constant polynomials $f \in \mathbb{C}[X]$ have roots?
Problem: Roots are very hard to find explicitly
Idea: Use $\pi(\text{disc with hole}) = \pi(S^1) \cong \mathbb{Z}$

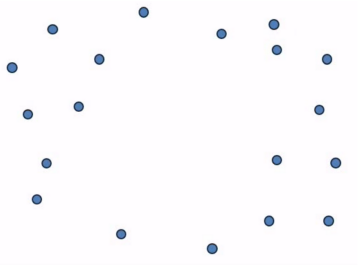
(a) If f has no root, then one gets loops $\gamma_R(\theta) = f(re^{i\Theta})/|f(re^{i\Theta})|$ in S^1

(b) γ_R gives an element of $\pi(S^1) \cong \mathbb{Z}$ for all R; use this to contradict (a)



Application two – topological data analysis

Life. Data has shape | Topology. Shape matters



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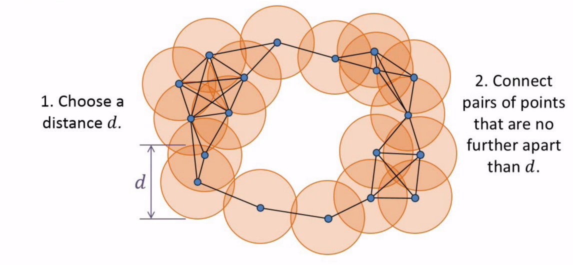

- Draw a circle of radius d around your data
- Two circles intersect $\Rightarrow$ draw a line; three circles intersect $\Rightarrow$ draw a triangle; $k+1$ circles intersect $\Rightarrow$ draw a k simplex
- How this changes for varying d is captured by persistent homology

Thank you for your attention!

I hope that was of some help.